BEHAVIOUR OF PARAMETERS OF NONLINEAR PRODUCTION FUNCTIONS USING CLASSICAL AND BAYESIAN APPROACHES

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ABSTRACT

Most production functions are often nonlinear in parameters and difficult to linearise. The major shortcomings of the classical approach in estimating such parameters are varying complex inter- parametric relationship, parameters estimates and their standard errors taking infeasible values in the process of iteration. Even when the initial values are carefully selected, the parameters of such models remain difficult to estimate often yielding ambiguous results owing to their specification in the models. This work therefore, proposed Bayesian estimators (BE) for estimating the parameters of nonlinear production function which ameliorate the problems of Classical Approach.

The Cobb-Douglas (CD) and Constant Elasticity of Substitution (CES) production functions are nonlinear in parameters. The CD production function with Additive Error (CDAE), CD Multiplicative Error (CDME) and (CES) were considered. Three possible specifications of Returns to Scale (RS) parameters were also investigated for CD namely: Constant RS (CRS, $\beta_1 + \beta_2 = 1,$), Increasing RS (IRS, $\beta_1 + \beta_2 > 1$) and Decreasing RS (DRS, $\beta_1 + \beta_2 < 1$), where β_1 and β_2 are the output elasticities of Capital and Labour, respectively. For CES, the behaviour of the substitution parameter ($\tau = -0.5$, 0 and 0.5) was chosen such that the Elasticity of Substitution (σ) was ($\frac{1}{1+\tau}$). The three BE: CDAE, CDME and CES were obtained using independent Normal Gamma prior. The posterior estimates was obtained using metropolis- within- Gibbs sampler. The performance of Bayesian approach (BA) and CA was compared using a Monte Carlo study with sample sizes (N = 50, 100, 150, 250 and 500) each in 10,000 iterations, considering the minimum Numerical Standard Error (NSE) or Standard Error (SE) as the criteria for BE and CA, respectively.

The derived BE were $p\left(\hat{\beta}/y^{*},h\right)\sim N\left(\overline{\beta},\overline{V}\right)$ and $p\left(h/y^{*},\hat{\beta}\right)\sim G\left(\overline{s}^{-2},\overline{v}\right)$ for CDAE, $p\left(\hat{\beta}/y^{**},h\right)\sim N\left(\overline{\beta},\overline{V}\right)$ and $p\left(h/y^{**},\hat{\beta}\right)\sim G\left(\overline{s}^{-2},\overline{v}\right)$ for CDME and $p\left(\lambda/y^{*},h\right)\sim N\left(\overline{\lambda},\overline{V}\right)$ and $p\left(h/y^{*},\lambda\right)\sim G\left(\overline{s}^{-2},\overline{v}\right)$ for CES. For CA, the estimates of $\hat{\beta}_1,\hat{\beta}_2,SE\left(\hat{\beta}_1\right)$ and $SE\left(\hat{\beta}_2\right)$ of CDAE and CDME were 0.841800, 0.148060, 0.014130, 0.007060 and 0.702530, 0.186370, 0.118600, 0.070790 for CRS, respectively. Those for IRS were 0.916189, 0.201102, 0.017134, 0.008889 and 0.816900, 0.419520, 0.128530, 0.091390, respectively. For DRS, they were 0.521822, 0.126701, 0.014130, 0.008448 and 0.618900, 0.179160, 0.109110, 0.068120, respectively. However, for BA, the estimates of $\hat{\beta}_1,\hat{\beta}_2,NSE\left(\hat{\beta}_1\right)$ and $NSE\left(\hat{\beta}_2\right)$ of CDAE and CDME were 0.759519, 0.132379, 0.000120, 0.000116 and 0.845861, 0.176499, 0.000114, 0.000103 for CRS, respectively. Those for IRS were 0.897875, 0.204861, 0.000118, 0.000114 and 0.894150, 0.227022, 0.000116, 0.000102, respectively. For DRS, they were 0.68244, 0.170446, 0.000121, 0.000114 and 0.644050, 0.143532, 0.000116, 0.000101, respectively. For CA, the CES estimates of σ and SE when $\sigma < 1$ were 0.014610, 2221.7626, when $\sigma = 1$ were -0.206960, 190.593100 and when $\sigma > 1$ were 0.995520, 0.849600, while the estimates of BA for σ and NSE when $\sigma < 1$ were 0.893575, 0.000817, when $\sigma = 1$ were 0.503699, 0.001117 and when $\sigma > 1$ were 1.462442, 0.001215. Thus, BE performed better than CA in all the specifications of production functions with minimum value in NSE.

The Bayesian estimators outperformed classical approach for the production functions considered, thus making them more appropriate in handling nonlinear production functions regardless of the error specification.

Keywords: Constant elasticity of substitution, Cobb-Douglas production function, Metropolis-within-Gibbs, Returns to scale parameters, Numerical standard error.

Word count: 450

DEDICATION

This work is dedicated to the glory of my Lord and Saviour, JESUS CHRIST, for His name be praised forever. Amen

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Finally, and above all, I return glory to the Almighty God for His Mercy toward me, His faithfulness and blessings to the sons of men, forever I will remain grateful. Thank you Lord!!!.

CERTIFICATION

I certify that Mrs Adesina, Oluwaseun Ayobami carried out this work in the Department of Statistics, Faculty of Science, University of Ibadan, Ibadan, Nigeria under my supervision.

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CHAPTER ONE

1.1 Introduction

A production function expresses the relationship between the physical output and capital and labour inputs or factors of production. It can be defined as a mathematical expression which describes the maximum level of output attainable from any given sets of inputs in aproduction process. This function is a useful tool in many fields of economic analysis. It deals with the analysis of economic growth and the determination of optimum patterns of international or interregional trade and provides a framework for determining the change in output attributable to change in units of factor inputs and also facilitates the study of the behaviour of relative factor shares. The behavioral relations among technical change, factor shares, the elasticity of substitution between factors of production and economic growth have been spelt out theoretically by Hicks (1932) in his well-known Theory of Wages.

The study of production functions has received more attention, in recent time, than most other aspects of quantitative economics. The existing volume of literature on this topic makes it a fairly burdensome task for current researchers in this area to justify their deliberations. However, because most of the major theoretical developments in the study of production functions have resulted from empirical observation, each investigation is often a potential source of theoretical advancement.

Theoretically the study of production functions started with the conceptualization of quantities of factors of production- labour, capital, etc. determining the value of output. This was later followed by the use of Euler's theorem on homogeneous functions, the analysis of returns to scale and of factor substitution.

A very outstanding empirical development in the study of production functions took place when Cobb and Douglas (1928) fitted the function later named after them to actual data. Since then many versions of empirical and theoretical production functions have been developed to analyse the economic growth at the micro and macro levels.

The two- factor production function can be stated implicitly as

$$y = f(X_1, X_2) \tag{1.1}$$

where X_1 and X_2 are capital and labour inputs respectively used in producing y units of output.

The Cobb- Douglas (C-D) production function is deterministically defined as:

$$y = \beta_0 X_1^{\beta_1} X_2^{\beta_2} \tag{1.2}$$

where X_1 denotes the capital input, X_2 is the labour input, β_0 is the efficiency parameter and y is the output. The generalized form of the C-D production function is defined as

$$F(x_1,...,x_n) = \beta_0 X_1^{\beta_1} \cdots X_n^{\beta_n}, \tag{1.3}$$

where $x_i > 0$ ($i =$ 1, $\cdots$,n) , β_0 is a positive constant and $\beta_1, \cdots, \beta_n$ are nonzero constants.

The earliest empirical measures of returns to scale parameter were provided by Cobb and Douglas (1928) in their pioneering work mentioned above. The assumption of constant returns to scale incorporated into their model was later queried by Durand (1937), who introduced the idea of variable returns. The variable returns to scale (VRS) has not been of much practical application because the assumption of constant returns to scale facilitates the assumption that output is just exhausted by the two or more factor inputs in perfectly competitive markets. Also, possible inadequacies of tests to determine the significance of departures of $\hat{\beta}_1 + \hat{\beta}_2$ from unity when fitting:

$$\log \hat{y} = \log \hat{\beta}_0 + \log \hat{\beta}_1 + \log \hat{\beta}_2 \tag{1.4}$$

have tended to induce the acceptance as evidence of constant returns to scale, any $\hat{\beta}_1 + \hat{\beta}_2$ which is close to unity on either side. In addition to its use in estimating the scale parameter, the C-D function has also been found useful in a number of other empirical contexts.

At the aggregate level, Zarembka (1966) found it useful in explaining pattern of international trade, Hallam (1991) in studying technological epochs, and Evenson and Gollin (2003)in identifying the sources of observed growth in agricultural productivity.

At less aggregate levels, evidence of increasing returns to scale was found with this function by Shaiara and Md. Shahidul (2016), Klein (1953) and by Griliches and Ringstad (1971).

The apparently overwhelming evidence of increasing returns to scale in underdeveloped economies, perhaps, reflect the small- scale nature of the observed producing units. At such scale of operation, equi-proportional adjustments in factor inputs are more easily undertaken.

The C-D production function assumed the elasticity ofsubstitution to be unity. The concept of a Constant Elasticity of substitution (CES) production function was developed by Stanford group Arrow, Chenery, Minhas, and Solow (1961) as a generalization of theC-D function which allows for any non-negative constant elasticity of substitution.
Thus, using the CES production function the explicit deterministic form is given as:

$$y = \gamma \left\{ \delta X_1^{-\tau} + (1-\delta) X_2^{-\tau} \right\}^{-\frac{v}{\tau}} \tag{1.5}$$

where y is the output, γ is the efficiency parameter, δ, $(0 < \delta < 1)$ determine factor shares, τ is the substitution parameter and v is the scale parameter.Equation (1.5) above embodies the constant elasticity of substitution $\sigma = (1+\tau)^{-1}$, hence for $\tau \to 0$ and elasticity of substitution $(\sigma) \to 1$.

Both the C-D and CES production function are nonlinear. However, when a multiplicative error term is used to estimate the C-D function it is intrinsically linear and this has been the popular version of C-D for most empirical studies however, when the additive error term is used to estimate the C-D function it is intrinsically nonlinear. On the other hand, the CES production function is intrinsically nonlinear whether the error term is additive or multiplicative. This has posed some estimation problems for the CES production function. In addition to nonlinear iterative methods, Kmenta (1967) provided a linearized version of this function in the neighbourhood of $\tau \to 0$. This study uses the Gauss Newton method which is an iterative nonlinear procedure to estimate the parameters of the CES production function and the Kmenta approximation of CES in the neighbourhood of $\tau \to 0$ to form the distribution of the likelihood for a Bayesian method.

The intrinsically nonlinear additive-error based C-D function is usually estimated in the frequentist approach using Gauss Newton iterative method. This was applied in the study. Earlier Isaac D. et.al (2011) investigated the consequences and the seriousness of mis-specifying the error term in the presence of multicollinearity. Essi (2000) showed that the consequence is more serious when a multiplicative error plagued data set is fitted with an additive error based model than vice versa. Md. Moyazzem et.al (2012, 2015) selected the appropriate Cobb-Douglas production model for measuring the production function of some industries and compared the Cobb-Douglas production function with additive error to the model with multiplicative error term. They found that Cobb-Douglas production function with additive error performed better.

This work compared the performance of three frequentist estimates of the multiplicative- error- based CES (MCES), multiplicative – error- based C-D (M C-D) and additive error- based C-D (A C-D) on the one hand with corresponding Bayesian estimates of these three functions, the critical values of measures of returns to scale i.e, $\beta_1 + \beta_2 = 1$, $\beta_1 + \beta_2 > 1$, and $\beta_1 + \beta_2 < 1$, and the elasticity of substitution $\sigma = 1$, $\sigma > 1$, and $\sigma < 1$, were used to compare the performances of the two categories of estimators.

The basic idea of classical nonlinear regression is the same as that of linear regression which is characterized by the fact that the prediction equation depends nonlinearly on one or more unknown parameters, whereas linear regression is often used for building a purely empirical model and it usually arises when there are physical reasons for believing that the relationship between the response and the predictors follows a particular functional form.

Instead of following the sampling theory approach of presenting a point estimate and its corresponding standard error, the Bayesian approach summarizes information about an unknown parameter in terms of a probability density function which describes how likely (subjective probabilistic) different values of the parameter are to be estimated, Griffiths et al. (1993). The likelihood principle led one towards a Bayesian approach since all approaches based on classical criteria invalidate this principle Berger and Wolpert (1988) compared with a true likelihood approach which is difficult to calibrate Royall (1997). In addition, another appealing characteristic is that the Bayesian approach to inference may be derived via decision theory. In Bayesian econometrics, error terms ε_i are assumed to be independently and normally distributed with mean zero and unknown variance σ^2 that is, $\varepsilon_i \sim N(0_N, h^{-1}N)$; where $h = \sigma^{-2}$ is a precision. All the elements of $X's$ are fixed (i.e. not random variables) or, if they are random variables, they are independent of all elements ε_i with probability density function $p(X \mid \beta)$, where β is a vector of parameters that does not include any of the other parameters in the model.

Bayesian statistics offers a rationalist theory of personality beliefs in the contexts of uncertainty, with the central aim of characterizing how an individual should act in order to avoid certain kinds of undesirable behavioral inconsistencies. Zellner (1971) came up with a publication on Bayesian econometrics that eventually gained popularity, and has subsequently been utilized in many applications Koop (1994). A basic point in Bayesian statistics is that all the uncertainty that we might have about a phenomenon should be described by means of probability. Statistics is based on applying the law of probability to statistical inference, all unknown quantities which are contained in a probability model for the observed data are considered to be random variables, let $\theta = (\theta_1, \ldots, \theta_p)$ denote all the unknown parameters of the model, and $y = (y_1, \ldots, y_n)$ the vector of observed data. Inference is made through the posterior probability distribution of θ, after observing y, $p(\theta \mid y) \alpha\, p(y \mid \theta) p(\theta)$. The term $p(\theta \mid y)$ is referred to as the Posterior density (this summarizes all we know about θ after seeing the data, $p(y \mid \theta)$ as likelihood function (is the density of the data conditional on the parameter of the model), and $p(\theta)$ as the prior density, the prior does not depend on the data(it represents the probability beliefs for θ before observing the data y).

Bayesian econometrics has enjoyed increasing popularity in many fields, owing to its generally-recognized desirable characteristics which provide a formal mechanism for prior information about an unknown parameter. Since economists frequently have prior information about the parameters in economic relationships, this characteristic is particularly appealing! There has been dramatic growth in the development and application of Bayesian inference in statistics. One reason for the dramatic growth in Bayesian modeling is the availability of computational algorithms to compute the range of integrals that are necessary in a Bayesian posterior analysis owing to the speed of modern computers; it is now possible to use the Bayesian paradigm to fit very complex models that cannot be fitted by alternative Frequentist methods. This makes the application of Bayesian econometrics the perceived need for numerical integration.

A common way to present results from a Bayesian investigation is to provide graphs of the marginal posterior probability density functions for each of the unknown parameters of interest. The learning process is solved through the application of probability rules: where one simply has to compute the conditional probability of the event of interest, given the experimental information. The chief competitor to Bayesian econometrics, often called frequentist econometrics, specifies that θ is not a random variable. However, Bayesian econometrics is based on a subjective view of probability, which argues that the uncertainty about any unknown can be expressed using the rules of probability. In order to present information in this way, it is necessary to derive marginal posterior probability density functions for each of the unknown parameter from the joint posterior probability density function for all the parameters. This process means that unwanted parameters must be integrated out of the joint posterior probability density function. If analytical integration is impossible which makes numerical integration impractical, some other solutions have to be found. The solution has led to an exposure to in the use of Markov Chain Monte Carlo (MCMC) technique. These techniques provide a method of drawing observations from the joint posterior probability density function Cowles and Carling (1996). Once some observations have been drawn, they can be used to make estimated marginal posterior probability density functions as accurate as we like, by drawing as many observations as are required. Marginal posterior probability density functions are not the only means of presenting information that utilizes MCMC-estimated integrals. Posterior means and standard deviations which are the Bayesian counterparts of sampling theory point estimates and standard errors often take the form of intractable integrals. These quantifiers can be readily estimated using the sample means and standard deviations of the MCMC-generated observations. There are two main MCMC techniques used in non- linear problems, Gibbs sampler, and the Metropolis-Hastings algorithm. The following researchers, Albert and Chib (1993), Chib and Greenberg (1995), Chib et al (1998), Tanner (1996) and Gilks et al. (1996) have worked with and appreciated the wide variety of Bayesian approaches using simulation techniques.

1.2 Bayesian Econometrics

Bayesian Econometrics seeks the combination or addition of Bayesian statistics in relevant ways to models and phenomena of interest to economists. It is based on a few simple rules of probability. What Econometricians wish to do is to estimate parameters of a model compare different models or obtain predictions from a model, which all involve a few general principles of probability, which are applied over and over again in different settings. Probability in Bayesian approach is one of interpretations of the concept of probability, in contrast to interpreting probability as frequency or propensity of some phenomenon, this probability is a quantity that is assigned to represent a state of knowledge, or a state of belief. In this view, a probability is assigned to a hypothesis, whereas under frequentist inference, hypothesis is typically tested without being assigned a probability. The interpretation of probability can be seen as an extension of propositional logic that enables reasoning with hypotheses, (the propositions whose truth or falsity is uncertain); the probability belongs to the category of evidential probabilities which is used to evaluate the probability of the hypothesis. The Bayesian probabilist specifies some prior probabilities, which are updated in the light of new and relevant data

There are two views on Bayesian probability that interpret the probability concept in different ways. According to the objectivist view, the rule of Bayesian statistics can be justified by requirements of rationality and consistency and interpreted as an extension of logic, while to subjectivist view, probability quantifies “a personal belief”.

Bayesians begin by writing down a joint distribution of all quantities under consideration (except known constants). The OLS quantities known under sampling are denoted by an N-dimensional vector y, and the remaining unknown quantities by the K-dimensional vector $\theta \in \theta \subseteq RK$, which is an inequality restriction where R is a known N x K matrix, unless noted, Bayesian Econometrics treats the Econometric Model parameters θ as otherwise continuous random variables. Working in terms of densities, consider for a variable y and θ of events.

$$P(y,\theta) = P(\theta)P(y \mid \theta) = P(y)P(\theta \mid y) \tag{1.6}$$

where $P(y,\theta)$ is the Joint probability density of both events y and θ , $P(\theta)$ is the prior density and $P(\theta \mid y)$ is the posterior density. Viewing $P(y \mid \theta)$ as a function of y for known θ, any function proportional to it is referred to as a likelihood function. Denote the likelihood function as $L(\theta)$. Unless noted otherwise, work with $L(\theta) = P(\theta \mid y)$ and thus include the integrating constant for $y \mid \theta$ in the description of the likelihood above, Note that linear regression model presupposes linear relationship between a dependent variable and explanatory variables, that is;

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon \tag{1.7}$$

where y is the response variable, $\beta = (\beta_1, \ldots, \beta_p)^T$ is a vector of regression parameters to be estimated and ε are the random errors which are assumed to be uncorrelated with mean zero and constant variance i.e $\varepsilon \sim N(0,1)$. Non-linearity arises in many diverse ways in Bayesian econometric applications. Perhaps the simplest and best known case of non-linearity in econometrics is that which arises as the observed variables in a linear regression model are transformed to take account of the first-order auto-regression of the error terms.

1.3 Characteristics of the Cobb-Douglas Production Function.

The Cobb-Douglas production function has the most important assumption of Constant Returns to Scale (CRS) – any proportional increase in both inputs results in an equi-proportional increase in output; that is, double both LABOUR (β_2) and CAPITAL (

β_1) inputs and you get double the Y real output. The CRS property occurs because the sum of the exponents on the LABOUR (β_2) and CAPITAL (β_1) input variables sum to one. The main advantage of these assumptions is simplicity. However these assumptions seem broadly consistent with empirical evidence at the macro level. The unit elasticity assumption is consistent with the relative constancy of nominal factor shares. In more general forms of this production function, the fractional exponents on the input variables could sum to less than one (decreasing returns to scale) or sum to greater than one (increasing returns to scale or economies of scale)Burnside et al.(1995) for econometric evidence.

1.4 Characteristics of the CES Production Function

The CES function includes three special cases; for $\tau \to 0$, σ goes to 1 then the CES turns to the Cobb-Douglas form; for $\tau \to \infty$, σ approaches 0 then CES turns to the Leontief production function; and for $\tau \to -1, \sigma$ goes infinity then CES yields to a linear function if v equals 1. However, the parameters of CES function are nonlinear in nature and it is analytically difficult to linearize and cannot be estimated with the familiar linear techniques. Hence, the CES function is approximated via "Kmenta approximation" Kmenta (1967) and by this the techniques of linear estimation can be applied. The CES function assumes that the Elasticities of Substitution among two inputs are the same. The functional form proposed by Uzawa (1962) has constant Allen Uzawa elasticities of substitution and the functional form proposed by McFadden (1963) has constant Hicks-McFadden Elasticities of Substitution.

The inputs CES functions proposed by Uzawa (1962) with McFadden (1963) impose strict conditions on the values for the Elasticities of Substitution and are less useful for empirical applications. Therefore, Sato (1967) proposed a family of two-level nested CES functions to have two levels of CES functions, where each of the inputs of an upper-level CES function might be replaced by the dependent variable of a lower-level CES function. These functions increased in popularity especially in the field of macro-econometrics, where input factors needed further differentiation Griliches (1969), Krusell et. al (2000), and Pandey (2008)

1.5 Aim and Objectives

1.5.1 Aim

This study is aimed at investigating the behaviour of three models, namely; The Cobb-Douglas production function with additive error term, The Cobb-Douglas production function with multiplicative error term and The Constant Elasticity of Substitution (CES) of production function in the macroeconomics production function using the Bayesian and Frequentist approaches.

1.5.2 Objectives of the Study.

The choice of Bayesian and Frequentist approach in a non-linear context in this study was to achieve the following objectives:

- To estimate the parameters of the C-D production functions using the Bayesian and Frequentist approaches
- To investigate the behaviour of the parameters of the multiplicative and additive-error- based C-D production functions under alterative assumptions of the scale parameter using the Bayesian and frequentist approaches
- To estimate the parameters of the multiplicative-error-based CES production function using the Bayesian and frequentist approaches.
- To investigate the effects (on the other parameters) of varying values of τ in the estimation of the Multiplicative CES production function.

1.6 Statement of Problem

In a general linear model, when the posterior is analytical in nature it is easy to compute the marginal of y , even with priors which preclude the availability of analytical results and also the posterior estimate. But in the Cobb- Douglas and CES production function relating an output, y, to inputs $X's$ (non-linear) model with an independent Normal-Gamma prior, the posterior does not have a known form (analytical form) from which samples should be drawn, hence the need to introduce a very important class of simulating technique called Metropolis-Hastings Algorithms; to obtain the estimation of posterior distribution which is the major focus of this study.

1.7 Justification of Study

In classical approach, to estimate the parameters of the nonlinear model there is usually the challenge of having varying complex inter parametric relationship, parameters estimates and/or their standard errors taking infeasible values in the process of iteration and convergence to local minimum (initial- value-dependent convergence). Even when the initial values are carefully selected, the parameters of non- linear models remained difficult to estimate often yielding ambiguous results owing to the way the parameters and error terms are specified in the models due to the factors around the selection of initial values (i.e. if the choice of the initial value is very far from unknown true values, there is possibility of attaining a convergence to local minimum or no convergence at all).

However, the Bayesian approach gives the opportunity of using past knowledge (prior) as a guide towards the choice of initial values because a researcher may have little or no prior knowledge about a field of investigation, also, to outline or portray an alternative method with the use of posterior simulation technique (Metropolis-within-Gibbs) which removes the effect of initial values through a process called "burn-in" and aid a quick or fast convergence.

1.8 Organization of the Thesis

This chapter gives the brief introduction on macroeconomic production functions which describes a systematic relationship between inputs and output in an economy. Bayesian econometrics encompasses the various ideologies of objective and subjective views of the literature on which the Bayes theory operates. Relevant literature on Cobb-Douglas and CES as well as the theoretical and methodological reviews by classical and Bayesian researchers on Cobb-Douglas and CES production function are well stated in Chapter 2. Chapter 3 gives a detailed description of the methodology used in the study (both the Classical and Bayesian approaches). A nonlinear regression model in Bayesian approach with an independent normal-gamma prior resulted in a truncated posterior for both the Cobb-Douglas & CES production function and the methodologies are explicitly illustrated to show how the estimates were obtained by using the posterior simulation technique called a Metropolis Within-Gibbs Algorithm. The results of the analysis and interpretation are presented in Chapter 4, while the summary and conclusion of the work has been clearly stated in Chapter 5.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In this chapter, a brief explanation and definition of the concepts that relate to nonlinear production functions both in CES and Cobb-Douglas within our study are given. Different methods employed are discussed on how to estimate the parameters of production functions in the literature using the classical and Bayesian approach, based on the specification of error terms in the models.

2.2 History of Bayesian Revolution in Econometrics

Rev. Thomas Bayes (1702-1761), a Mathematician (known for the introduction of the Bayes theorem) who first used probability inductively and established a mathematical basis for probability inference (a means of calculating from the number of times an event has not occurred, the probability that it will occur in future trials, he set down his findings on probability in,'Essay' towards solving a problem in the Doctrine of Chances(1763), published posthumously in the Philosophical Transactions of the Royal Statistical Society of London.

The only works he was known to have published in his lifetime are Divine Benevolence or an Attempt to prove that the principal end of the divine providence and Government is the happiness of his creatures (1731) and An introduction to the Doctrine of Fluxions and a defense of the Mathematicians against the objectives of the author of the Analyst (1736) which encountered attacks by Bishop Berkeley on the logical foundations of Newton's calculus, he was a Presbyterian minister in Tunbridge Wells from 1731 , son of the Rev. Joshua Bayes, a non-conformist minister. It is thought that his election to the royal statistical society might have been based on a tract of 1736 in which Bayes defended the views and philosophy of Sir Isaac Newton. A notebook of his exists and includes a method of finding the time and place of conjunction of two planets, notes on weights and measures, a method of differentiation, and logarithms.

2.3 Preliminaries: Basic Idea About Bayesian Inference

Bayesian statistics is certainly one of the fastest developing areas in Statistics. With the advent of highly intensive computer technology, it is now a highly practical methodology for addressing many important high dimensional decision problems as well as being underpinned by a sound mathematical foundation. It is essentially helpful when some of the components of uncertainty have sparsely collected data associated with them, so that expert judgment need to be incorporated. It simply forces the analyst to consider historical datasets. In the analysis of real data in any field, the information is perfect on the phenomenon of interest.

A basic point in Bayesian statistics is that all the uncertainty that we might have on a phenomenon should be described by means of probability Petris (2010). In the Bayesian viewpoint, probability has a subjective interpretation, being a way of formalizing the incomplete information that the researcher has about the events of interest. The technique commonly used for obtaining the posterior distribution (described below) is well-known as Bayes theorem which is the foundation of Bayesian methodology.

2.4 Bayes' Theorem

Bayes' theorem calculates the posterior distribution as proportional to the product of a prior distribution and the likelihood function. Given two events A and B, probability rules says that the joint probability of A and B occurring is given by

$$P(A \cap B) = P(A \mid B)P(B) \qquad (2.1)$$

where $P(A \mid B)$ is the probability of A occurring conditional on B having occurred and $P(B)$ is the marginal probability of B, reversing the roles of A and B, find the expression for joint probability of A and B;

$$P(A \cap B) = P(B \mid A)P(A) \qquad (2.2)$$

Rearrange the expression in equation (2.1) and (2.2) to give Bayes' rule on which Bayesian Econometrics is based and states that if the sample space S is partitioned into mutually exclusive and exhaustive events $B_1, B_2, \cdots, B_K$ and A is another event defined on the sample space, then

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} \tag{2.3}$$

where $P(A) = \sum_{i=1}^{k} P(A \mid B_i)P(B_i)$

The application of Bayes theorem using data is of interest to researchers. Economists usually work with the model which depends on parameters. In this model, interest centered on the coefficient in the regression and the researcher is interested in estimating these coefficients. Let y be a vector or matrix of data and θ be a vector or matrix which contains the parameters for a model which seeks to explain y to estimate θ based on the data, Bayes' rule is then applied to obtain

$$P(\theta \mid y) = \frac{P(y \mid \theta)P(\theta)}{P(y)}, \ P(y) > 0 \tag{2.4}$$

$$\equiv \frac{P(y \mid \theta)P(\theta)}{\int P(y \mid \theta)P(\theta)d\theta} \text{ for continuous}$$

and

$$\frac{P(y \mid \theta)P(\theta)}{\sum_{i=0}^{k} P(y \mid \theta)P(\theta)} \text{ for discrete}$$

The importance of this theorem in Bayesian statistics is in the interpretation and scope of the inputs of the two sides of the equation in the theorem, and in the role of Bayes theorem in the subjective learning of process.

2.5 The Prior Specification

Priors are generally believed to be information at hand before seeing the data and random variables can be thought of as variables that take on values with specifiedprobability. Parameters are not random but fixed and unknown quantities in a classical approach, however anything we are not certain about in Bayesian can be thought of as random to which probability distribution is assigned, known as prior information. Prior represents a belief about parameter values. Bayesian analysts seek a balance between prior information in a form of expert knowledge or belief and evidence from data at hand. Choice of prior is a sensitive and controversial aspect in any Bayesian analysis. Very serious attention should be geared towards the choice of this prior, which can be created using a number of methods. These are information (past or no) and principle of prior such as symmetry or maximizing entropy given constraints (Jeffrey's principle).There are two types of prior; informative and non-informative

Non informative is also known as flat prior. This is a type of prior where there is no sufficient information about the parameter. Researchers see this prior as objective because of the flexibility of its choice. The term non-informative is misleading, since all priors contain some information, so such priors are generally better referred to as vague or diffuse. The subjectivity problem is then handled by assigning equal probability to all states of parameter space.Jeffry (1961) developed another form of non informative prior called Jeffry priorthat satisfies the local uniformity property and does not change much over the region in which the likelihood is significant and does not assume large values outside the range. This prior is for large samples posterior densities for parameters thatassume a normal form with a posterior mean equal to the maximum likelihood estimate and posterior covariance matrix equal to inverse of the estimated Fisher information matrix.Many researchers have developed on the use of non-informative prior using Jeffrey's principle for Bayesian justification in Bernardo and Smith (1994), Berger (1985), Box and Tiao (1973), Gelman et al (1995), Press (1989), Robert and Casella (2004).

Kass and Wasserma (1996) interpreted noninformative priors as formal representations of ignorance and no objectivity when there is no sufficient information to define priors. These noninformative priors are chosen by public agreement on standard unit of measurement. Yang (1995) developed hybrid Markov chain sampling scheme using noninformative priors to analyze a Random Coefficient Regression (RCR) model and showed that the corresponding posteriors are proper when the number of full rank design matrices are greater than or equal to twice the number of regression coefficient parameters plus 1 and that the posterior means for all parameters exist if one more additional full rank design matrix is available. Andrew et. al (2008) proposed a new prior distribution for nonhierarchical logistic regression model constructed by first scaling of nonbinary variables by placing independent student t prior distribution on the coefficient in which, in the simplest setting, is a longer -tailed version of the distribution attained by assuming one- half additional success and one-half additional failure on logistic regression. The result shows that the Cauchy class of prior distribution outperformed the existing Gaussian and Laplace priors.

Informative prior: This is a type of prior where there is information about parameters. It brings about evidence about the parameters concerned from different sources. This prior is not dominated by the likelihood and that has an impact on the posterior distribution. In other words, if a prior distribution dominates the likelihood, it is clearly an informative prior. Yahya et at (2014) compared the classical ordinary least square (OLS) regression technique with the Bayesian conjugate normal linear regression method when the data satisfy all the necessary assumptions of OLS technique, and found from Monte Carlo study that the OLS estimator is as good as the Bayesian estimator in terms of the closeness of their estimated parameters to the true values. On the contrary, using the criteria of the mean square errors of parameters' estimates and other performance indices, the results showed that Bayesian estimator is more efficient, more consistent and relatively more stable than the classical least squares method even when the sample data satisfy all the necessary assumptions of the OLS method. James (2018) specified informative prior distribution for the probabilities in a multinomial for flexible multivariate distributions to reduce that of the marginal distributions, the ordering of the probabilities and the conditional and unconditional built bivariate copulas stacked in a tree structure based on D-vine. This avoids issues of under parameterization associated with conjugate priors such as the Dirichlet distribution and the direct specification of a positive definite correlation matrix. Ulrich (2012) derived measures of prior sensitivity and prior informativeness that account for the high dimensional interaction between prior and likelihood information, which turns out to be a simple function of the posterior and prior covariance matrices and thus makes it entirely straightforward to compute the measures from the output of standard posterior samplers.

2.6 The Likelihood Function

The likelihood function is defined as the joint probability density function for all the data conditional on the unknown parameters. It is simply the name given to the model applied to the data e.g Normal, Gamma or Log-normal models. In classical framework, all estimation and inference are based solely on observed data.

The likelihood contains all the information about the model parameters that can be learnt from the data. Let $X^n = (X_1, X_2, ..., X_n)$ have joint density $P(X^n;\theta) = P(X_1, X_2 ..., X_n;\theta)$ where $\theta \in \Theta$. the likelihood function $L : \theta \to [0,\infty]$ is defined by $L(\theta) \equiv L(\theta; x^n) = p(x^n;\theta)$ where x^n is fixed and θ varies in Θ. the likelihood function is a function of θ and it is not a probability density function. If the data are independent and identically distributed, then the likelihood

$$L(\theta) = \prod_{i=1}^{n} p(x_i;\theta) \tag{2.5}$$

which is defined only to a constant of proportionality. The likelihood function is used to generate estimators like the maximum likelihood estimator and it is a key ingredient in Bayesian inference.

2.7 The Posterior Distribution

The posterior analysis describes the probability distribution of all model parameters conditional on the observed data and some prior knowledge. The posterior distribution has two components: a likelihood that includes informationabout model parameters based on the observed data and a prior that includes prior information about model parameters.The likelihood and prior are combined using the Bayes rule to produce the posterior distribution.

posterior $\propto$ likelihood x prior

In practice, a convenient way of directly proceeding to posterior inference is to make it numerically analytical. However, this is not the case, the posterior distribution of model parameters is numerically intractable (Tanner and Wong, 1987). This implies that it is impossible to derive the closed form of posterior estimates. The posterior distribution is rarely available explicitly and needs to be estimated via simulations. MCMC sampling can be used to simulate potentially very complex posterior models with an arbitrary level of precision. MCMC methods for simulating Bayesian models are often demanding in terms of specifying an efficient sampling algorithm and verifying the convergence of the algorithm to the desired posterior distribution

2.8 Bayesian Algorithms

An algorithm is a step by step systematic method for carrying out a procedure for solving scientific problems, usually with the condition that the procedure terminates at a point Atallah (1998). The process of relating an algorithm to an input to obtain an output is known as computation.

A crucial pioneering advance in Bayesian econometrics is owing to the advancement of effective and efficient Monte Carlo algorithms like the Gibbs sampling algorithm considered by Tanner and Wong (1987) and Gelfand et. al. (1990). Since the arrival of computational revolution, the matter of computing power has been deepest in Econometric analysis, but it is even more pronounced to Bayesian econometrics when the Monte Carlo algorithms are applied.

In a surprising way, Markov chain Monte Carlo (MCMC) algorithms, mostly the Metropolis- Hasting algorithm Metropolis et.al. (1953, Hasting (1970) and the Gibbs sampling algorithms Gelfand et. al. (1990) have transpired as highly popular tools for the econometric models anaysis.

Bayesian analysis entails the evaluation of complex and often high- dimensional integrals in order to obtain posterior distributions for unknown parameters of interest in the model. As considered in this work, most macroeconomic variables are assumed to follow a Markov chain. A Markov chain is a mathematical model for stochastic systems where the description of the present state fully expresses all the information that could influence evolution of the process through a probabilistic process Jiang (2009).

MCMC algorithms require large amounts of computationally intensive simulations and nontrivial are often needed for certain extensions, such as the one in this work. The Bayesian algorithm requires that a prior distribution is specified for each parameter in the estimation procedure to produce posterior distributions. While MCMC allows an enormous expansion of the class of candidate models for a given dataset, they also suffer from a well- known and potentially serious problem: it is often difficult to decide when it is safe to terminate them and conclude their "convergence" Zellner (2009).

2.9 Existing Studies

The Cobb Douglas production function was first introduced by Charles W. Cobb and Paul H. Douglas (1928), although anticipated by Knut Wicksell (1851-1926) to report the production output which determined the amount of Labour and the invested amount of Capital in different industries of the world as the involved universal law of production; on the contrary Cobb and Douglas (1928) tested against this proposition by revealing that the coefficient of two factors (Labour and Capital) considered were not constant over time or between the same sectors of the economy. These two factors of Cobb-Douglas aggregate production function were discussed in the seminal contribution and which was preferred in the growth theory of specification Solow (1956). The popularity of Cobb-Douglas is rarely surprising in given its analytical features and tractability.

Nevertheless, several researchers have expressed concern about the empirical relevance of this specification. Among others, it is the only linearly homogenous production function with a constant elasticity of substitution in which each factor share of income is constant over time. The CES production function due to Arrow et al. (1961) reported that estimates are consistent and Kmenta (1967) used the same estimation procedures applicable to the generalized version of the CES function which is restricted to the case of constant returns to scale, and concluded that the estimates are consistent if the input variables are non-stochastic or, if stochastic, disturbance is independent in the production function.

Simon and Michael (2007) demonstrated that the common practice of using Cobb-Douglas or Leontief production functions in economic model should be rejected for the majority of sectors. In their study, results compared to standard linear estimations using Kmenta approximations, non-linear estimation techniques perform significantly better. Moreover, during the time period considered, no significant change in input substitutability took place over time. Despite the central role of substitution of elasticities within the theoretical and empirical framework of applied simulation, the current situation of elasticities is still unsatisfying and although the problem of substitution parameter that leads to elasticities of substitution has been acknowledged but then persisted for a while (Dawkins et al. , 2001).

Frazer (2002) reexamined the original data sets of time-series of an aggregate production function used by Douglas to undertake and establish the existence of an aggregate production function; he particularly paid attention to whether the data offered reasonable support for the law of production in which the various statistical methods were used to analysed and confirm the data set, found that the data sets yielded the results that support the assertions of Douglas.

Duffy and Papageorgiou (2000) formally questioned the empirical relevance of Cobb-Douglas where they used a panel of 82 countries over 28 years and estimated a two-input constant elasticity of substitution (CES) production function specification of which Cobb-Douglas is a special case. For the entire sample of countries they strongly rejected the Cobb-Douglas specification over the more flexible CES specification.

Moreover, Miller (2008) studied based on macroeconomic production in which he examined the margin strengths and weakness of the functional forms (Cobb-Douglas and CES production) and suggested that the form is appropriate for use in macroeconomic, to forecast income shares under given policies; then recommended to continue using Cobb-Douglas in spite of theoretical concerns because it appears that additional cost and parameters uncertainties from the use of the CES are not outweighed by its benefits.

Houthakker (1955-1956) showed how to aggregate out a Cobb-Douglas production from underlying Leontief production functions when their coefficients are jointly Pareto distributed. In support of the result obtained, Jones (2005) partially built on this to show that a global Cobb-Douglas production function can arise from general constant returns to scale production function.

Leon-Ledesma and Satchi (2011) developed a technology choice model to show how a parameter choice in (a specific CES production function with an elasticity of substitution below unity) a Cobb-Douglas reduced form production function can be established, and reported that the class of production functions derived are consistent with a balanced growth path even in the presence of capital-augmenting technical progress. However, there are constraints that arise in the extreme choices of capital intensity since all tasks cannot be performed by either capital or labor, and the choice of intensity through time by costs.

In contrast, Acemoglu and Zilibotti (2001) demonstrated how a Cobb-Douglas production function after endogenous technology adjustment leads to a CES production with a higher than unity substitution elasticity to show that the differences in the supply of skills create a mismatch between the requirements of these technologies and the skills of Less Developed Countries(LDC) workers. Mismatch led to sizeable differences in total factor productivity and output per worker in favor of the cross-industry productivity patterns predicted.

Haykel (2002) proposed to estimate the elasticity of substitution resulting from the CES (Constant Elasticity of Substitution) production function which has been commonly used by modellers in construction their Computable General Equilibrium models, in their studied they use statistical data to out the analysis on elasticity of substitution by employing the method of Least square which produced two factors of production and reported that the estimation led to a specification of the production functions which appeared as Cobb-Douglas, since the elasticity of substitution was equal to 1 and this took the form of CES.

Nakamura and Nakamura (2008) confirmed Acemoglu and Zilibotti's work by using a more general functional form for the intermediates' productivities to show how a general CES production with elasticity above unity can arise from an underlying Cobb-Douglas technology. In their specification, two primary input factors are differentiated over a unit interval of intermediate inputs. Each input used just one of the primary factors. Productivity of the intermediate inputs depends on their position in the interval through a specific functional form. Choosing which primary factor is used for each intermediate input in a profit maximizing way results in a CES production function with an elasticity of exactly 2 in the two primary factors.

Cantore et.al (2015) estimated an elasticity of substitution between capital and labour using Bayesian-Maximum-Likelihood methods to show that the result is driven by the implied fluctuations of factor shares under the CES specification. The improved CES model performed than CD under an imperfect information assumption with elasticity well behaved when it is below unity. Owing to problems of quantitative measurement of some key parameters in production function models at constant economies of scale in production and constant share of labor in total population level of performance becomes dependent on the presence of additional factors and technology Nikolay et. al (2016); hence the increase in population leads to slower productivity growth, if population growth will not affect the development of other factors. If population growth stimulates the development of other factors of production and/or technology, labor productivity growth will accelerate or slow down depending on the ratio of the level of influence of positive and negative effects.

Werf (2007) showed that empirical analysis summary quantitative models for climate policy modeling differed in the production structure used and in the sizes of the elasticities of substitution. The elasticities were rejected when equal to one in favour of considerably smaller values, with lower elasticities and with factor-specific technological change, some climate policy models found a bigger effect of endogenous technological change on mitigating the costs of climate policy. Then estimated the parameters of two level CES production function with capital, labour and energy as inputs and revealed that nesting structure fits the data best.

Ashkan (2012) utilized Cobb-Douglas production function in construction crashing cost analysis and showed labor and equipment efficiencies have significant effects on total cost of a project. Harry(1998) considered the specification of a production function using crop-yield response to agricultural lime test several functions that are consistent with the theory by goodness-of fit tests, non-nested hypothesis tests and costs of misspecification and found that neither goodness-of-fit tests nor non-nested hypothesis tests provided a clear choice among the candidate functions; costs of misspecification provided some choice.Peter (2010) reported that the educational production function is approximately homogeneous of degree one in the productivity factors, in accordance with the Walberg's psychological theory of educational productivity based on the Cobb-Douglas economic production function theory.

However, after using improved time series data on labor, capital, output and factor payments, Berndt (1976) concluded that one cannot reject the aggregate production function which is Cobb-Douglas. A number of works have challenged his result, as Berndt assumed that all technical change was Hicks neutral. Labor augmenting technical change causes estimates of the elasticity to be biased towards one. At the aggregate level, the capital share of cost is constant while the capital-labor ratio is rising, leading regressions to conclude that the production function is Cobb Douglas.

Rainer and Harald (2000) examined the inconsistencies and controversies in the use of CES production functions in growth models in which they showed all the variants of CES functions are consistently specified and also find that a higher elasticity of substitution leads to a higher steady state and makes the occurrence of permanent growth more probable and reported the speed of convergence has effect on higher elasticity of substitution which depends on the relative scarcity of the factors of production.

Henningsen and Henningsen (2012) considered the difficulty in the estimation of the non-linear Constant Elasticity of Scale (CES) function due to convergence problems and infeasible results which frequently arises from an incompatible objective function by large uniform areas, the interruption of the CES function when the elasticity of substitution is one and the significant rounding errors when the elasticity of substitution is close to one and they suggested three solutions that ease these problems which make the results to be more dependable and easily accepted.

Jürgen (2014) addressed the relationship between technical change and the elasticity of substitution between factors of production and showed how the elasticity within a CES production setting can change due to technical change. However, if labor augmenting productivity increases at the same rate as the capital-labor ratio, any elasticity is consistent with a stable aggregate capital share of cost.

Antras (2004) and Klump et. al. (2007) both controlled for labor augmenting technical change through time trends (implying exponential growth in labor augmenting productivity) or other parametric functional forms and estimate the elasticity to be 0.8 and 0.6 and concluded that the for the elasticity to be below, equal to, or greater than one depends on the type of technical change applied.

Kamel and Maha (2015) compared the degrees of capital and labour substitution in a developing country in order to change the tradition of some modelers and presented the estimates of Elasticity of substitution caused by two production functions in which the nonlinear least squares method was used to estimate the substitution elasticity in each sector of CES function and SURE method to estimate the translog function and The results indicate a strong similarity between the elasticity substitution estimates for each function, which are significantly below the unity.

Nadia and Olimpiu (2015) described the analysis of the stationary and the dynamic caseof the Kmenta approach for estimating the parameters of the CES production function by using the least square method and found that the bestresults are obtained with the dynamic model in which the deviations between theempirical and adjusted values are the lowest compared to static model

Hideo and Koki Kyo (2015) proposed a new approach in analysis of factor augmenting technical change based on a constant elasticity of substitution (CES) production function. Miguel and Mathan (2011) showed that allowing firms a choice of CES production techniques through the distribution parameter between capital and labor can result in a new class of production functions that are consistent with a balanced growth path even in the presence of capital augmenting technical progress which produces short-run capital-labor complementarity but yields a long-run unit elasticity of substitution.

Bartlesman and Doms (2000) estimated trends in labor and capital augmenting technical change in which smoothness priors are introduced and Bayesian linear models are constructed to examine the technical changes in Taiwan and South Korea at the macroeconomic level and revealed that the Bayesian approach can capture the movements of technical change more rigorously than conventional approaches. Industrial Organization economists are investigating how productivity differences across firms are related to market structure, and more generally what causes differences in productivity across firms .

Many macroeconomic models assume that productivity shocks cause business cycle fluctuations. Steenkamp (2016) revealed that negative capital-augmenting technical change in several industries weighed on productivity in New Zealand based on Constant Elasticity of Substitution (CES) production functions that permit varying assumptions about factor augmentation and also allows for industry-specific values of the elasticity of substitution between inputs.

In view of this Abidemi (2010) examined productivity in the banking sector by way of estimating two major production functions known in the economic literature and reported that result obtained from the Ordinary Least Squares (OLS) estimates for substitution parameters α and β (substitution parameters for capital and Labour, respectively) confirmed the a priori expectation that the duo of α and β are positive values of less than one. The substitution parameters in the Constant Elasticity of Substitution Production Function were equally positive, which supports the theory.

Adetunji et al (2012) examined Capital-Labour Substitution and Banking Sector Performance in Nigeria between 1960-2008 and in furtherance, studied the Nigerian economy probably characterize constant returns to scale for the sample period between 1990 to 2009. The Capital/Labour ratio increased by 1 percent, on average, Labour productivity went up by about 1 Percent .The results of the models obtained when compared with the work of Liedholm (1964) and Osagie and Odaro (1975) gave satisfactory results in terms of goodness of fit. This then showed that the Cobb-Douglas Production Function (CDPF) gives a better explanation of the aggregate production process in the banking industry in Nigeria for the period studied.

A broader literature in macroeconomics and labor economics focuses on estimating the elasticity of substitution between labor and capital, using a variety of different techniques. Most of the estimates of the macro elasticity of substitution perform better than the micro elasticity of substitution. The early debate on the elasticity of substitution focused on differences between time series and cross section estimates of the elasticity. These early papers used relations between labor productivity and wages as capital data was unavailable. The cross section estimates based on two digit sector aggregates across states or countries and local area wage differences, had high elasticities, often above one. However, the cross-section estimates had severe biases due to differences in labor quality and industry composition across areas with which time series estimates were significantly below one.

Henningson and Henningson (2011) demonstrated and described several approaches by replicating some existing estimation methods of the Cobb Douglas and CES production function with two explanatory variables, also extended it to nested CES functions with three and four explanatory variables, for which an R package called micEconCES was developed to handle these Classical Nonlinear estimation of the CES production function. Findings revealed that the corresponding sum of squares error of optimization produced the least minimum value which shows the validity of the methods developed.

Devesh (2011) identified the Labour Capital Elasticity of substitution using local market wage variation. He used the Cobb-Douglas specification and CES production function with Labour augmenting production that causes dispersion in the micro data. He concluded that a CES production function over imposed Cobb- Douglas specification. Miguel and Mathan (2011) reaveled that the new class of production functions (CES type) derived are consistent with a balanced growth path even in the presence of capital-augmenting technical progress while allowing firms to make a choice of capital intensity in the production process as well as capital and labor. In view of this, Jakub and Jakub (2015) showed that estimates are consistent which implied that the elasticity of substitution between capital and labor has remained relatively stable, at about 0.8–0.9, from 1948 to the 1980s, followed by a period of secular decline in post-war US economy by generalize the normalized Constant Elasticity of Substitution (CES) production function by allowing the elasticity of substitution to vary isoelastically.

Martinez (2012) proposed a Bayesian Markov Chain Monte Carlo estimation of the capital-Labour substitution elasticity in developing countries through prior elicitation and concluded that the Bayesian estimator of the Capital-Labour substitution elasticity was theory consistent, and can be used to properly calibrate computable general equilibrium models. Diamond et al (1978) chose the correct functional form in the presence of biased technological change is challenging because it is impossible to determine whether the evolution of factor shares and factor ratios over time is due to elasticity effects or technological bias unless a priori assumptions about the structure of technological change are imposed.

Pereira (2003)reported that Elasticity of Substitution is non unitary and changing over time; this study showed that Monto Carlo simulation for usual CES production did not give realiable estimates for the substitution parameter and that the use of Cobb - Douglas production function hid the roles of Elasticity of Substitution not only as a source of income in output but also as a source of technical change. In addition with a CES to calculate Total Factor Productivity(TFP) across counrtries, the variance of TFP is lower than that of a Cobb-Douglas that made income difference across countries diminished

Leon- Ledesma et. al. (2010) employed Monte Carlo sampling techniques and applied it to the generalized form of the Kmenta approximation to highlight the empirical advantages of normalization, their study has revealed that the direct estimation of the non-linear CES does not alleviate identification problems, and also reported the weak results because of the involvement of numerical complexity in estimating the non-linear CES, more so, the identification problems can be Substantial when using many conventional approaches and found that the normalization offered several theoretical consistent which improves the empirical identification

The use of MCMC simulation is to let the parameters perform a random walk in parameter space according to a Markov chain set up in such a way that its stationary distribution is the posterior distribution. Hastings (1970) proposed a useful algorithm for setting up the Markov chain which is the Metropolis-Hastings (MH) algorithm, and supported by Chen et al.(2000) that it is very convenient and suitable for problem. Geweke (1992) proposed the evaluation accuracy of sampling-based approaches to calculating posterior moments and also suggested a diagnostic based on the intuition that if a sufficiently large number of draws have been taken, the estimate of $g(\theta)$ based on the first half of the draws should be the same as the estimate based on the last half.

2.10 Related literature

A number of approaches have been suggested for estimating nonlinear production due to sensitivity of classical approach to choice of starting values and that the parameters often failed to converge. Early work by Hideo and Koki (2013). Luoma and Luoto(2011) used Bayesian approach to estimate the parameter of normalized constant elasticity of substitution function with factor augmenting technical progress and found that posterior simuation of parameters behaved fairly well, in addition the results indicated that in the long run the parameters for the elasticity of substation and capital income share are intimately linked to the share of capital augmenting technological progress converged to zero which led to upward biased estimates

Koop and Steel (2001) described a Bayesian method for efficiency analysis using a special type of model called stochastic frontier models with cross-sectional data and a log-linear frontier. The posterior estimates were obtained using the Bayesian Posterior simulation techniques known as the Gibbs sampler. It was observed by the authors that in the case of the nonlinear frontier the posterior simulation methods were complicated.

The simple production frontier model is of the form:

$$Y_i = f(X_i, \beta) \qquad (2.6)$$

Where Y_i is the output of firm i and is produced using a vector of inputs, $X_i, (i = 1, \dots N)$. in practice the actual output of a firm may fall below the maximum possible. The form of interest in many applications is the measurement of inefficiency of the actual output, which is as follows:

$$Y_i = f(X_i, \beta)\tau_i \qquad (2.7)$$

The additional of measurement error makes the frontier stochastic; hence the linear (Cobb Douglas or Log-linear) model is given as:

$$y_i = x_i\beta + v_i - z_i \qquad (2.8)$$

Where; x_i is defined as a $1 \times (K+1)$ vector (e.g. $X_i = (1, L_i, K_i)$) in the case of a Cobb-Douglas frontier with two inputs L & k.

$\beta = (\beta_0,\ldots,\beta_k)'$, $y_i = \ln(Y_i)$, $v_i = \ln(\varsigma_i)$, $z_i = -\ln(\tau_i)$ and x_i is the counterpart of X_i with inputs transformed to log. τ_i is referred to as inefficiency and since $0 \le \tau_i \le 1$.

For the Bayesian Inference, the following assumptions were made about v_i and z_i for $i = 1\ldots N$.

1. $P(v_i / h^{-1}) = f_N^{'}(v_i / 0, h^{-1})$ and the v_is are independent
2. $v_i \& z_i$ are independent of one another for all i and l.
3. $P(z_i / \lambda^{-1}) = f_N^{'}(z_i / 1, \lambda^{-1})$ and the $z_i s$ are independent

Ritter and Simor (1997) shows that the use of very flexible one-sided distributions for z_i such that the unrestricted Gamma may result in a problem of weak identification. Intuitively, if z_i is left too flexible, then the intercept minus z_i can come to look much like v_i and it may become virtually impossible to distinguish between these two components with small data sets.

In addition, van den Broeck et al (1994) found the Exponential model to be the least sensitive to changes in prior assumptions in a study of the most commonly used models.

The likelihood function is defined as:

$$L(y;\theta) = \prod_{i-1}^{N} P(y_i / x_i;\theta)$$

Where β can be obtained using OLS estimation method from $\hat{\beta} = (x'x)^{-1}x'(y+z)$, where $y + z = x\beta + v$ which requires the derivation of $p(y / x_i, \theta) = \int p(y_i / x_i, z_i, \theta) p(z_i / \theta) dz_i$. this is done in Jondrow, Lovell, Materor and Schmidt (1982) for the Exponential model and in van dan Broeck et al (1994) for a wider class of inefficiency distribution.

Bayesian Inference can be carried out using a posterior simulator which generates draws from the posterior, $p(\theta / y, x)$. in this case, Gibbs sampling with data augmentation in a natural choice for a posterior simulator.

The Prior employed in this paper are:

A Uniform prior for β is $p(\beta) \propto I(E)$

Where; $I(E)$ is the indicator function for the economic regularity conditions for the other parameter, they assumed Gamma priors:

$$p(h) = f_G(h / a_h, b_h)$$

and $$p(\lambda^{-1}) = f_G(\lambda^{-1} / a_\lambda, b_\lambda)$$

setting $a_h = 0$ and $b_h = 0$ obtained $p(h) \propto h^{-1}$, the usual non-informative prior for the error precision in Nonlinear regression model.

Also, setting $a_\lambda = 1$ and $b_\lambda = -\ln(\tau^*)$ yields a relatively non-informative prior.

The Full Posterior $p(z / y, x_i, \beta, h, \lambda^{-1}) \propto f_N^N(z / x\beta - y - h\lambda_{\iota_N}{}^{-1}, h^{-1}I_N)\prod_{i=1}^{N} I(z_i \geq 0).$

The Nonlinear model (CES Production function) considered for this model is given as:

$$y_i = f(x_i, \beta) + v_i - z_i \tag{2.9}$$

It was observed that the Posterior simulator for everything other thing except β is almost identical to the one given by the linear model.

The same prior were used while the conditional Posterior for β is more complicated, having the form:

$p(\beta / y, x, z, h, \lambda^{-1}) \propto \exp(-\frac{h}{2}\sum_{i=1}^{N}(y_i = f(x_i, \beta) + z_i)^2)p(\beta)$ on the other hand, if no convenient approximating density can be found, a random walk chain –Metropolis-Hasting algorithm might be a good choice Chib and Greenberg (1995).

Martinez (2012) provides a Bayesian MC^2 estimation of the capital-labor substitution elasticity in developing countries.

The paper defined the Nonlinear model (CES Production function) as:

$$q = A\left[\rho k^{\frac{\sigma-1}{\sigma}} + (1-\rho)l^{\left(\frac{\sigma-1}{\sigma}\right)}\right], \tag{2.10}$$

Where; q is real aspect, A is Hicks-neutral technological shifter, $\rho \in [0,1]$ is a distribution parameter and $\sigma \in [0,\infty]$ is the elasticity of substitution between capital and labor. Two first order conditions emerged from profit maximization by forms in a competitive framework; the labor demand obtained from these conditions equals.

$$\frac{q}{l} = \left(\frac{\omega}{\rho}\right)^{\sigma} (\rho)^{-\sigma} A^{1-\sigma}, \tag{2.11}$$

this equation can be log-linearized,

$$y = \theta_0 + \theta_1 x_i, \tag{2.12}$$

for $y = \ln(ql)$, $x = \ln(\omega / p)$, w/p real wages and $\theta_0 = -\sigma \ln \rho + (1-\sigma)\ln A$, $\theta_1 = \sigma$ as in Cicowiez(2011).

Adding an error term and time or cross section subscripts, the equation above becomes a linear regression model. OLS and 2SLS was used to carry out the classical estimation of parameters.

Using the Bayesian (method) Approach; the likelihood correspondents are stated as:

$y \sim N(X\theta, s^2 I_n)$ where; $X = [1_n, (x_1, \ldots, x_t)'], \theta = [\theta_0, \theta_1], y = (y_1, \ldots, y_t)'$

With prior distributions; $\underline{\theta} \sim N(\theta_0, \beta_0), s^2 \sim IG(\alpha_0 / 2, \delta_0 / 2)$. As a consequence of these assumption

$$\overline{\theta} / s^2, y \sim N(\underline{\theta}, \beta_1),$$

where; $\beta_1 = [s^{-1} X'X + \beta_0^{-1}]^{-1}$

$\underline{\theta} = \beta_1 [s^{-1} X'y + \beta_0^{-1} \theta_o]^{-1}$

and $\alpha_1 = \alpha_0 + n$

$$\delta_1 = \delta_0 + (y - X\theta)'(y - X\theta)$$

Since both conditional posterior distributions are standard, a Markov Chain Monte Carlo sampler can be used to find the Posterior distribution of (θ, s^2);

(a) Let $s^{2(0)}$ be a starting value of s^2.

(b) At the gth iteration.

$$\theta^{(g)} \sim N(\theta^{-(g)}, \beta_1^{(g)})$$

$$s^{2(g)} \sim IG(\alpha/2, \delta^{(g)}/2)$$

Where;

$$\beta_1^{(g)} = [s^{-2(g-1)} X'X + \beta_0^{-1}]^{-1}$$

$$\overline{\theta}^{(g)} = \beta_1^{(g)}[s^{-2(g-1)} X'y + \beta_0^{-1}\theta_0],$$

$$\delta^{(g)} = \delta_0 + (y - X\theta^{(g)})'(y - X\theta^{(g)})$$

With repetition of the Gibbs sampling (b) until $g = B + G$, where B is the burn-in sample and G is the desired sample size and calculated the values of $\theta^{(g)}$ and $s^{2(g)}, g = B+1, \ldots, B+G$ and simulate the Posterior distribution of θ and s^2.

Koop (2003) showed the estimation of two Nonlinear Regression models, a situation whereby the first Cobb-Douglas production function is Intrinsically linear while the second Nonlinear model known as the CES production function proved to be Non intrinsically linear through the log-linear method of linearization.

The Nonlinear models are stated as:

For the Cobb-Douglas:

$$y = \alpha_1 x_2^{\beta_2}, \ldots, x_k^{\beta_k} \tag{2.13}$$

in which if the log of both sides is taken ith an addition of an error term, we obtain a regression model:

$$\ln(y_i) = \beta_1 + \beta_2 \ln(x_{i2}) + \ldots + \beta_k \ln(x_{ik})$$

Where; $\beta_1 = \ln(\alpha_1)$, which is now linear in logs of the dependent and explanatory variables and with this small differences all the techniques of the OLS estimation methods apply.

The CES production function is of the form:

$$y_i = \left(\sum_{j=1}^{k} \gamma_j x_{ij}^{\gamma_{k+1}} \right)^{\frac{1}{\gamma_{k+1}}} \quad (2.14)$$

The Bayesian inference in regression model was considered where the explanatory variable enter in an intrinsically nonlinear way. He focused the empirical illustration on using the CES production function for this case in the nonlinear regression model of the form:

$$y_i = \left(\sum_{j=1}^{k} \gamma_j x_{ij}^{\gamma_{k+1}} \right)^{\frac{1}{\gamma_{k+1}}} + \varepsilon_i \quad (2.15)$$

Where; ε & y are N-vectors stalking the errors and observation of the dependent variable respectively and let X be an N X k matrix stacking the observation of the K explanatory variables, he made the standard assumptions that:

1. ε is $N(0_N, h^{-1}I_N)$
2. All elements of X are either fixed or they are random variables, they are independent of ε with a probability density function $p(X / \lambda)$.

Where; λ is a vector of parameter that does not include any of the other parameter in the model.

Let the general nonlinear regression model:

$y_i = f(X_i, \gamma) + \varepsilon_i$ where; X_i is the ith row of $X_i, f(.)$ is a function which depends upon X_i and a vector of parameter γ.

The model can be written as: $y_i = \gamma_i + (\gamma_2 x_{i1}^{\gamma_4} + \gamma_3 x_{i2}^{\gamma_4}) \dfrac{1}{\gamma_4} + \varepsilon_i$ simplified as

$$y = f(X,\gamma) + \varepsilon \tag{2.16}$$

Where; $f(X,\gamma)$ is now an N-vector of function with i[th] element given by $f(X_i,\gamma)$. The exact implementation of the Posterior simulation algorithm will depend upon the form of $f(.)$.

The likelihood function is expressed as shown below using the definition of the multivariate normal density:

$$P(y/\gamma,h) = \frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{\exp\left[-\frac{h}{2}\{y-f(X,\gamma)\}'\{y-f(X,\gamma)\}\right]\right\}$$

with the linear regression model, we are able to write this expression in terms of OLS quantities.

The Prior choice depends upon what $f(.)$ is and how γ is interpreted, by which Koop (2003) used a non-informative prior of the form:

$$p(\gamma,h) \propto \frac{1}{h},$$

A Uniform prior for γ , $\underline{\gamma} = (1,1,1,1)'$, $\underline{V} = 0.25I_4$, $\underline{v} = 12$ and $\ln(h)$.

The Posterior density is then obtained as:

$$P(\gamma,h/y) \propto P(\gamma,h).\frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{\exp\left[-\frac{h}{2}\{y-f(X,\gamma)\}'\{y-f(X,\gamma)\}\right]\right\}$$

In general, there is no way to simplify this expression which will depend upon the precise forms for $P(\gamma,h)$ and $f(.)$ and doesn't take the form of any well-known density. The resulting marginal posterior for γ is

$$P(\gamma/y) \propto \left[\{y-f(X,\gamma)\}'\{y-f(X,\gamma)\}\right]^{-\frac{N}{2}}.$$

In the case where $f(.)$ was linear, this expression could be rearranged to be out in the form of the kernel of a t-distribution, but here it does not take any convenient form.

The Computational Techniques: Metropolis Hastings Algorithm

The lack of analytical results relating to the posterior suggests that a posterior simulator is required. For some forms of $f(.)$ it may be possible to derive a Gibbs sampler. Inother words, if $g(.)$ is our function of interest, we can obtain an estimate of $E[g(\theta / y)]$, which we label $\widehat{g}_s$, by simply averaging the draws in the familiar way:

$$\widehat{g}_s = \frac{1}{s}\sum_{r=1}^{s} g(\theta^{(s)})$$

The Metropolis-Hastings algorithm always takes the following form:

Step 0: choose a starting value, $\theta^{(0)}$

Step 1: Take a candidate draws, θ^{*} form the candidate generating density $q(\theta^{(s-1)},\theta)$

Step 2: Calculate an acceptance probability $\alpha(\theta^{(s-1)},\theta^{*})$

Step 3: Set $\theta^{(s)} = \theta^{*}$ with probability $\alpha(\theta^{(s-1)},\theta^{*})$ and set $\theta^{(s)} = \theta^{(s-1)}$ with probability $1-\alpha(\theta^{(s-1)},\theta^{*})$

Step 4: Repeat step 1, 2 and 3 …, S times.

Step 5: Take the average of the S draws $g(\theta^{(s-1)}),\ldots,g(\theta^{(s)})$.

These steps will yield an estimate of $E[g(\theta / y)]$ for any function of interest. It turns out that acceptance probability has the form:

$$\alpha(\theta^{(s-1)},\theta^{*}) = \min\left[\frac{p(\theta=\theta^{*} / y)q(\theta^{*};\theta=\theta^{(s-1)})}{p(\theta=\theta^{(s-1)} / y)q(\theta^{(s-1)};\theta=\theta^{*})},1\right]$$

The Metropolis-Hastings Algorithm is of three types:

1. The Independent Metropolis-Hastings algorithm: the independent chain Metropolis-Hasting-Algorithm uses a candidate generating density which is independent across draws. i.e. $q(\theta^{(s-1)},\theta) = q^{*}(\theta)$ and candidate draws does not depend upon $\theta^{(s-1)}$. Such an implication is useful where a convenient approximation exists to the posterior. This

convenient approximation can be used as the candidate generating density so that the acceptance probability simplifies to

$$\alpha(\theta^{(s-1)}, \theta^{*}) = \min\left[\frac{p(\theta = \theta^{*} / y)q(\theta = \theta^{(s-1)})}{p(\theta = \theta^{(s-1)} / y)q(\theta = \theta^{*})}, 1\right]$$

2. The Random Walk Chain Metropolis Hasting algorithm: this is useful when there is no good approximating density for the posterior, no attempt is made to approximate the posterior rather the candidate generating density is chosen to wander widely, taking draws proportionately in various regions of the posterior.

   The Random Walk Chain Metropolis Hastings algorithm generates candidates' draws according to $\theta^{*} = \theta^{(s-1)} + z$, the acceptance probability simplifies to $\alpha(\theta^{(s-1)}, \theta^{*}) = \min\left[\frac{p(\theta = \theta^{*} / y)}{p(\theta = \theta^{(s-1)} / y)}, 1\right]$

   It can be clearly seen that the random walk chain tends to move towards regions of higher posterior probability.

3. Metropolis within Gibbs: this provides a posterior simulator for $p(\theta / y)$, as illustrated below:

   $h(y, \gamma) \sim G(\bar{s}^{-2}, \bar{v})$, $\bar{v} = N + \underline{v}$, and $\bar{s}^{-2} = \frac{[y - f(X,\gamma)][y - f(X,\gamma)] + \underline{v}\underline{s}^{2}}{\bar{v}}$

   Note that: $p(\gamma / y, h) \propto p(\gamma, h / y)$ it can be seen that

$$p(\gamma / y, h) \propto \exp\left[-\frac{h}{2}\{y - f(X,\gamma)\}'\{y - f(X,\gamma)\}\right]$$

$$p(\gamma / y, h) \propto \exp\left[-\frac{h}{2}(\gamma - \underline{\gamma})'\underline{V}^{-1}(\gamma - \underline{\gamma})\right]$$

   this conditional posterior density does not have a form that can be directly drawn from in a simpler manner. Hence, the use of a Metropolis-Hasting algorithm for $p(\gamma / y, h)$ which combined the draws from the Gamma distribution which provides us with a Metropolis Within Gibbs algorithm.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

In this chapter, the methods used in estimating the parameters of the nonlinear production functions with the error specifications in Classical and Bayesian estimation with selection of prior using the Markov Chain Monte Carlo technique (MCMC). The Metropolis-Hasting algorithm and the data generating process (DGP) clearly stated for Production functions were also discussed.

3.1 Classical Approach

The nonlinear model is of the form

$$y_i = f(X_i, \beta) + u_i \qquad i = 1, 2, ..., N \tag{3.1}$$

Where y_i is response variable, $f(X_i, \beta)$ is the nonlinear form comprising of the explanatory variable and the coefficient of the model, and $u_i \sim N(0, \sigma^2)$, the error component with mean 0 and variance σ^2. The coefficients of explanatory variables X_i and u_i are to be estimated. The matrix form of the model in equation (3.1) can be expressed as;

$$Y = f(X, \beta) + U \tag{3.2}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \; X = \begin{bmatrix} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \\ \vdots & \vdots & \vdots \\ X_{1N} & X_{2N} & X_{3N} \end{bmatrix}, \; \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}, \text{ and } U = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$$

Where Y is an ($N \times 1$) vector of dependent variables, the parameter β is a $K \times 1$ vector, X is an $N \times K$ matrix of explanatory variables and U isan $N \times 1$ vector random variable representing the disturbance term, where N is the sample size and K is the number of parameters to be estimated.

The estimates of the nonlinear model in this study can be obtained by:

i. The Guass Newton method via Kmenta approximation method

i. The Gauss Newton Method (GNM)

The Gauss Newton method is one of the classical ways of estimating a nonlinear model, therefore, the universal steps of estimating any form of nonlinear model are stated below:

The Gauss Newton method begin by expanding

$$f(X_i, \beta) = f_i(\beta)$$

using Taylor's series up to the first derivative around a set of initial values, $\beta_j^{0\prime} = (\beta_0^0, \beta_1^0, \beta_2^0)$ and representing the required parameters appropriately,

Set $\lambda_j = \beta_j - \beta_j^0$

$Y_i^0 = f_i^0 = f(x_i, \lambda_j^0)$

And the set initial values given as

$\lambda_j^0 = \beta_0^0, \beta_1^0, \beta_2^0$.

Then the Gauss Newton method can be generalized as

$$Y_i - Y_i^0 = \sum \frac{\partial f(X_i, \beta^0)}{\partial \beta_j} \bigg|_{\beta = \beta^0} \left(\beta_j - \beta_j^0\right) + U \text{, for j = 0, 1, 2,…, J and i = 1, 2, 3, …, N} \quad (3.3)$$

Expressing out in terms of the parameters of the model, we obtain

$$Y_i - Y_i^0 = \frac{\partial f(X,\beta_0^{\ 0})}{\partial \beta_0}\left(\beta_0 - \beta_0^0\right) + \frac{\partial f(X,\beta_1^{\ 0})}{\partial \beta_1}\left(\beta_1 - \beta_1^0\right) + \cdots + \frac{\partial f(X,\beta_J^{\ 0})}{\partial \beta_J}\left(\beta_J - \beta_J^0\right) + U \tag{3.4}$$

Note that, $Y_i^0 = f_i^0 = f(X_{i1}, X_{i2}, \cdots, X_{ik}, \theta_0^0, \theta_1^0, \cdots, \theta_j^0)$ model with parameters, replaced by θ_j^0, $j = 1, \cdots, J$ and X's replaced by observed data

Let $$Z_{ij} = \frac{\partial f(X,\beta)}{\partial \beta_j}$$

and

$$\lambda_j = \beta_j - \beta_j^0$$

Therefore,

$$Y_i - Y_i^{\ 0} = Y_i - f_i^{\ 0} = \lambda_0^{\ 0} Z_{1i} + \lambda_1^{\ 0} Z_{2i} + \cdots + \lambda_J^{\ 0} Z_{Ni} + U \tag{3.5}$$

Then (3.4) can be written as

$$Y_i - Y_i^0 = Y_i - f_i^0 = \sum_{j=1}^{J} \lambda_j^0 Z_{ji} + U \tag{3.6}$$

Where equation (3.4) is the partial jth derivative of $f(X_i, \beta)$ with β's replaced by β_j^0. Equation (3.3) can be written as a linearized form of y_i in the neighbourhood of β_j.

For N-observations we have

$$\left.\begin{aligned} Y_1 - f_1^0 &= \lambda_1^0 Z_{11}^0 + \lambda_2^0 Z_{21}^0 + \ldots + \lambda_j^0 Z_{j1}^0 + U_1 \\ Y_2 - f_2^0 &= \lambda_1^0 Z_{12}^0 + \lambda_2^0 Z_{22}^0 + \ldots + \lambda_j^0 Z_{j2}^0 + U_2 \\ &\vdots \\ Y_N - f_N^0 &= \lambda_1^0 Z_{1N}^0 + \lambda_2^0 Z_{2N}^0 + \ldots + \lambda_j^0 Z_{jN}^0 + U_N \end{aligned}\right\} \tag{3.7}$$

i.e.

$$\begin{bmatrix} Y_1 - f_1^0 \\ Y_2 - f_2^0 \\ \vdots \\ Y_N - f_N^0 \end{bmatrix} = \begin{bmatrix} Z_{11}^0 & Z_{21}^0 & \cdots & Z_{J1}^0 \\ Z_{12}^0 & Z_{22}^0 & \cdots & Z_{J2}^0 \\ \vdots & \vdots & \cdots & \vdots \\ Z_{JN}^0 & Z_{JN}^0 & \cdots & Z_{JN}^0 \end{bmatrix} \begin{bmatrix} \lambda_0^0 \\ \lambda_1^0 \\ \vdots \\ \lambda_J^0 \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} \tag{3.8}$$

In this study, the above matrix equation can be simplified as;

$$Y - f^0 = Z^0 \cdot \lambda^0 + U \tag{3.9}$$

where

$Y - f_1^0$ is an $Nx1$ column vector

Z^0 is an $N\,x\,J$ matrix

λ^0 is an $J\,x\,1$ matrix, the modified equation parameters and

U represents a well behaved error component

The derivation represented by Z is as follow;

$$Z_{11} = \frac{\partial y}{\partial \beta_0}\left[f(X,\beta)\right]$$

$$Z_{21} = \frac{\partial y}{\partial \beta_1}\left[f(X,\beta)\right]$$

$$\vdots$$

$$Z_{JN} = \frac{\partial y}{\partial \beta_J}\left[f(X,\beta)\right]$$

Using the OLS method, we obtain the OLS estimate to be

$$\hat{\lambda} = \left(Z^{0\prime} Z^0\right)^{-1} Z^{0\prime}(D), \tag{3.10}$$

where; $D = Y - f^0$, Since, $\lambda_j^0 = \beta_j^1 - \beta_j^0$, the revised estimate of β_j is β_j^1.

Hence, $$\beta_j^1 = \hat{\lambda}_j^0 + \beta_j^0 \tag{3.11}$$

This is the end of the first iteration, to start the second iteration, we use the β_j^1 instead of β_j^0 and repeat all the processes above till the desired estimate is obtained.

Equation (3.2) can be minimized to obtain the sum of squares error by

$$S^0(\lambda) = \sum_{i=1}^{N}\left[Y_i - f^0 - \sum_j \lambda_j^{01} Z_{j1}{}^0\right]^2$$

$$S^0(\lambda) = \sum_{i=1}^{N}\left[D - \sum_j \lambda_j^{01} Z_{j1}{}^0\right]^2 \qquad (3.12)$$

Where; $S^0(\lambda)$ is the error sum of squares of the linearized model, therefore,

$$S^0(\lambda) = \left[Y_1 - f_1^0 - \sum_{j=1} \lambda_j{}^0 Z_{j1}{}^0\right]^2 + \left[Y_2 - f_2{}^0 - \sum_{j=1} \lambda_j{}^0 Z_{j2}{}^0\right]^2 + \cdots + \left[Y_N - f_N{}^0 - \sum_{j=1} \lambda_j{}^0 Z_{JN}{}^0\right]^2$$
(3.13)

Obtaining the *(r+1)^th^* iteration if $\left|(\beta^{r+1} - \beta^r)\middle|\beta^r\right| < \delta$, where δ is some predetermined magnitude

Recall

$\lambda_j^0 = \beta_j^1 - \beta_j^0$

where $\beta_j^1 = \hat{\lambda}_j^0 + \beta_j^0$ is the end of the first iteration, to start the second iteration, we use β_j^1, the revised estimate as the initial values now, then repeating all steps above and replacing the "0" superscripts by "1" will lead to another set of revised estimate β_j^2 which will mark the end of 2nd iteration.

Note: $\lambda_j^0 = \beta_j{}^1 - \beta_j{}^0$ *i.e.* $\beta_j{}^1 = \lambda_j^0 + \beta_j{}^0$ is a general rule to follow;

$\lambda_1^0 = \beta_1{}^1 - \beta_1{}^0$ *i.e.* $\beta_1{}^1 = \lambda_0^0 + \beta_1{}^0$ at the end of 1st iteration

$\lambda_2^0 = \beta_2{}^2 - \beta_2{}^1$ *i.e.* $\beta_2{}^2 = \lambda_1^0 + \beta_2{}^1$ at the end of 2nd iteration

$$\lambda_{j-1}^{0} = \beta_j^{\,r} - \beta_j^{\,r-1} \qquad i.e. \qquad \beta_j^{\,r} = \lambda_{j-1}^{0} + \beta_j^{\,r-1} \quad \text{at the end of r}^{\text{th}} \text{ iteration}$$

$$\vdots$$

$$\lambda_j^{0} = \beta_j^{\,r+1} - \beta_j^{\,r} \qquad i.e. \qquad \beta_j^{\,r+1} = \lambda^{r} + \beta_j^{\,r} \quad \text{at the end of (r+1)}^{\text{th}}$$

iteration

Hence; using equation (3.11) $\beta^{r+1} = \beta^{r} = \left[Z^{r\prime} Z^{r}\right]^{-1} Z^{r\prime}(Y - f^{r})$ are the least squares estimates obtained at the end of the (r+1) th iteration

$$\beta^{r+1} = \beta^{r} + \hat{\lambda}^{r}$$

$$\beta^{r+1} = \beta^{r} + \hat{\lambda}^{r}$$

$$\therefore \beta^{r} = \hat{\lambda}^{r} - \beta^{r+1} \text{ or } \beta^{r} = \beta^{r+1} - \hat{\lambda}^{r}$$

ii. **The Kmenta Approximation method**

The CES function is non-linear in parameters and cannot be linearized analytically; it is not possible to estimate it with the usual linear estimation techniques. Hence, the CES function is often approximated by the "Kmenta approximation" (Kmenta 1967) has a popularized and modified by Uebe (2000), Hoff (2004) and Henningson and Henningson (2010, 2011, 2012) which can be estimated by linear estimation techniques.

Derivation of the Kmenta Approximation

Consider the CES function;

$$y = \gamma \left[\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau} \right]^{-\frac{v}{\tau}} e^{u} \tag{3.14}$$

where, y is the response variable, γ determines the productivity, δ, the distribution parameter, τ, the substitution parameter, v, the scale parameter x_1 and x_2 are the explanatory variables. u is the well behaved error component (i.e. $N(0,\sigma^2)$)

Obtain the logarithm of equation (3.14) to have;

$$\ln y = \ln \gamma - \frac{v}{\tau} \ln \left[\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau} \right] + u \tag{3.15}$$

Let $$f(\tau) \equiv -\frac{v}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau}) \tag{3.16}$$

So that, $$\ln y = \ln \gamma + f(\tau) + u \tag{3.17}$$

Approximate the logarithm of the CES function by a first order Taylor series approximation and set $\tau = 0$

$$\ln y \approx \ln \gamma + f(0) + \tau f'(0) + u \tag{3.18}$$

Let $$g(\tau) \equiv \delta x_1^{-\tau} + (1-\delta) x_2^{-\tau} \tag{3.19}$$

Then, $$f(\tau) = -\frac{v}{\tau} \ln(g(\tau)) \tag{3.20}$$

Calculating the first partial derivative of $f(\tau)$ in equation (3.20) gives;

$$\frac{\partial f(\tau)}{\partial \tau} = f'(\tau) = \frac{v}{\tau^2} \ln(g(\tau)) - \frac{v}{\tau} \frac{g'(\tau)}{g(\tau)} \tag{3.21}$$

The first three derivatives of $g(\tau)$ are;

Recall, $$g(\tau) \equiv \delta x_1^{-\tau} + (1-\delta) x_2^{-\tau}$$

$$\frac{\partial g(\tau)}{\partial \tau} = g'(\tau) = -\delta x_1^{-\tau} \ln x_1 - (1-\delta) x_2^{-\tau} \ln x_2 \tag{3.22}$$

$$\frac{\partial^2 g(\tau)}{\partial \tau^2} = g''(\tau) = \delta x_1^{-\tau} (\ln x_1)^2 + (1-\delta) x_2^{-\tau} (\ln x_2)^2 \tag{3.23}$$

$$\frac{\partial^3 g(\tau)}{\partial \tau^3} = g'''(\tau) = -\delta x_1^{-\tau} (\ln x_1)^3 - (1-\delta) x_2^{-\tau} (\ln x_2)^3 \tag{3.24}$$

At the point of approximation, let $\tau = 0$, we have;

$$g(0) \equiv \delta x_1^{-(0)} + (1-\delta) x_2^{-(0)} \tag{3.25}$$

$$g'(0) = -\delta x_1^{-(0)} \ln x_1 - (1-\delta) x_2^{-(0)} \ln x_2 \tag{3.26}$$

$$g''(0) = \delta x_1^{-(0)} (\ln x_1)^2 + (1-\delta) x_2^{-(0)} (\ln x_2)^2 \tag{3.27}$$

$$g'''(0) = -\delta x_1^{-(0)}(\ln x_1)^3 - (1-\delta)x_2^{-(0)}(\ln x_2)^3 \quad (3.28)$$

This implies;

$$g(0) = \delta + 1 - \delta = 1 \quad (3.29)$$

$$g'(0) = -\delta \ln x_1 - (1-\delta)\ln x_2 \quad (3.30)$$

$$g''(0) = \delta(\ln x_1)^2 + (1-\delta)(\ln x_2)^2 \quad (3.31)$$

$$g'''(0) = -\delta(\ln x_1)^3 - (1-\delta)(\ln x_2)^3 \quad (3.32)$$

Also, the limit of $f(\tau)$ when $\tau = 0$ is;

$$f(0) = \lim_{\tau \to 0} f(\tau)$$

$$f(0) = \lim_{\tau \to 0} -v\frac{\ln(g(\tau))}{\tau}$$

$$f(0) = \lim_{\tau \to 0} -v\frac{\frac{g'(\tau)}{g(\tau)}}{1}$$

$$f(0) = v(\delta \ln x_1 + (1-\delta)\ln x_2) \quad (3.33)$$

The limit of $f'(\tau)$ when $\tau = 0$ is;

$$f'(0) = \lim_{\tau \to 0} f'(\tau)$$

$$f'(0) = \lim_{\tau \to 0}\left(\frac{v}{\tau^2}\ln(g(\tau)) - \frac{vg'(\tau)}{\tau g(\tau)}\right)$$

$$f'(0) = \lim_{\tau \to 0}\left(\frac{v\ln(g(\tau)) - \frac{v\tau g'(\tau)}{g(\tau)}}{\tau^2}\right)$$

$$f'(0) = \lim_{\tau \to 0}\left(\frac{\frac{vg'(\tau)}{g(\tau)} - \frac{vg'(\tau)}{g(\tau)} - v\tau\frac{g''(\tau)g(\tau) - (g'(\tau))^2}{(g(\tau))^2}}{2\tau}\right)$$

$$f'(0) = \lim_{\tau \to 0} -\frac{v}{2} \frac{g''(\tau) g(\tau) - (g'(\tau))^2}{(g(\tau))^2}$$

$$f'(0) = -\frac{v}{2} \frac{g''(0) g(0) - (g'(0))^2}{(g(0))^2}$$

$$f'(0) = -\frac{v}{2}\left(\delta (\ln x_1)^2 + (1-\delta)(\ln x_2)^2 - (-\delta \ln x_1 - (1-\delta)\ln x_2)^2\right)$$

$$f'(0) = -\frac{v}{2}\left(\delta (\ln x_1)^2 + (1-\delta)(\ln x_2)^2 - \delta^2 (\ln x_1)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2 - (1-\delta)^2 (\ln x_2)^2\right)$$

$$f'(0) = -\frac{v}{2}\left((\delta - \delta^2)(\ln x_1)^2 + ((1-\delta) - (1-\delta)^2)(\ln x_2)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2\right)$$

$$f'(0) = -\frac{v}{2}\left(\delta(1-\delta)(\ln x_1)^2 + (1-\delta)(1-(1-\delta))(\ln x_2)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2\right)$$

$$f'(0) = -\frac{v\delta(1-\delta)}{2}\left((\ln x_1)^2 + (\ln x_2)^2 - 2\ln x_1 \ln x_2\right)$$

$$f'(0) = -\frac{v\delta(1-\delta)}{2}\left(\ln x_1 - \ln x_2\right)^2 \qquad (3.34)$$

Substitute equation (3.33) for $f(0)$, and equation (3.34) for $f'(0)$

Recall from equation (3.18), the first order Taylor series approximation around $\tau = 0$ is;

$$\ln y \approx \ln \gamma + v\delta \ln x_1 + v(1-\delta)\ln x_2 - \frac{1}{2} v\tau\delta(1-\delta)(\ln x_1 - \ln x_2)^2 \qquad (3.35)$$

Derivatives with Respect to the Coefficients from equation (3.14)

Derivative with respect to "Gamma"

$$\frac{dy}{d\gamma} = [\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}]^{-\frac{v}{\tau}} e^u$$

$$\frac{dy}{d\gamma} = \exp\left(-\frac{v}{\tau}\ln[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}]\right)e^u$$

$$\frac{dy}{d\gamma} = \exp\left(f(\tau)\right)e^u$$

$$\frac{dy}{d\gamma} = \exp\left(f(0) + \tau f'(0)\right)e^{u} \tag{3.36}$$

Recall, $f(0)$ and $f'(0)$ from equation (3.33) and (3.34) respectively and substitute into equation (3.36) then;

$$\frac{dy}{d\gamma} = \exp\left(v\delta \ln x_1 + v(1-\delta)\ln x_2 - \frac{1}{2}v\tau\delta(1-\delta)(\ln x_1 - \ln x_2)^2\right)e^{u}$$

$$\frac{dy}{d\gamma} = x_1^{v\delta} x_2^{v(1-\delta)} \exp\left(-\frac{1}{2}v\tau\delta(1-\delta)(\ln x_1 - \ln x_2)^2\right)e^{u} \tag{3.37}$$

Derivative with respect to "Delta"

$$\frac{dy}{d\delta} = -\gamma\frac{v}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}-1}(x_1^{-\tau} - x_2^{-\tau})e^{u}$$

$$\frac{dy}{d\delta} = -\gamma v\frac{x_1^{-\tau} - x_2^{-\tau}}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}-1}e^{u} \tag{3.38}$$

Let $$f_\delta(\tau) = \frac{x_1^{-\tau} - x_2^{-\tau}}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}-1}$$

$$f_\delta(\tau) = \frac{x_1^{-\tau} - x_2^{-\tau}}{\tau}\exp\left(-\left(\frac{v}{\tau}+1\right)\ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)\right) \tag{3.39}$$

Equation (3.38) can be approximated using the first order Taylor series approximation of $f_\delta(\tau)$;

$$\frac{dy}{d\delta} = -\gamma v f_\delta(\tau)e^{u}$$

$$\frac{dy}{d\delta} \approx -\gamma v(f_\delta(0) + \tau f_\delta'(0))e^{u} \tag{3.40}$$

Define the helper functions $g_\delta(\tau)$ and $h_\delta(\tau)$

$$g_\delta(\tau) = \left(\frac{v}{\tau}+1\right)\ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)$$

$$g_\delta(\tau) = \left(\frac{v}{\tau}+1\right)\ln\left(g(\tau)\right) \tag{3.41}$$

$$h_\delta(\tau) = \frac{x_1^{-\tau} - x_2^{-\tau}}{\tau} \tag{3.42}$$

Take the first derivatives of equation (3.41) and (3.42) and subject $\tau = 0$;

$$g'_\delta(\tau) = -\frac{v}{\tau^2}\ln\left(g_\delta(\tau)\right) + \left(\frac{v}{\tau}+1\right)\frac{g'(\tau)}{g(\tau)} \tag{3.43}$$

$$h'_\delta(\tau) = \frac{-\tau(\ln x_1 x_1^{-\tau} - \ln x_2 x_2^{-\tau}) - x_1^{-\tau} + x_2^{-\tau}}{\tau^2} \tag{3.44}$$

Substituting into equation (3.39); $f_\delta(\tau) = h_\delta(\tau)\exp(-g_\delta(\tau))$ (3.45)

And differentiating using product rule, we obtained;

$$f_\delta{}'(\tau) = h'_\delta(\tau)\exp(-g_\delta(\tau)) - h_\delta(\tau)\exp(-g_\delta(\tau))g'_\delta(\tau) \tag{3.46}$$

Now taking the limits of $g_\delta(\tau)$, $g'_\delta(\tau)$, $h_\delta(\tau)$ and $h'_\delta(\tau)$ for $\tau = 0$ by

$$g_\delta(0) = \lim_{\tau\to 0} g_\delta(\tau)$$

$$g_\delta(0) = \lim_{\tau\to 0}\left(\left(\frac{v}{\tau}+1\right)\ln\left(g(\tau)\right)\right)$$

$$g_\delta(0) = \lim_{\tau\to 0}\frac{(v+\tau)\ln(g(\tau))}{\tau}$$

$$g_\delta(0) = \lim_{\tau\to 0}\frac{\ln(g(\tau)) + (v+\tau)\dfrac{g'(\tau)}{g(\tau)}}{1}$$

$$g_\delta(0) = \ln(g(0)) + v\frac{g'(0)}{g(0)}$$

$$g_\delta(0) = -v\delta\ln x_1 - v(1-\delta)\ln x_2 \tag{3.47}$$

Secondly, $g'_\delta(0) = \lim_{\tau\to 0}\dfrac{-v\ln(g(\tau)) + \tau(v+\tau)\dfrac{g'(\tau)}{g(\tau)}}{\tau^2}$

$$g'_\delta(0) = \lim_{\tau\to 0} \frac{-v\dfrac{g'(\tau)}{g(\tau)} + (v+\tau)\dfrac{g'(\tau)}{g(\tau)} + \tau\dfrac{g'(\tau)}{g(\tau)} + \tau(v+\tau)\dfrac{g''(\tau)g(\tau)-(g'(\tau))^2}{(g(\tau))^2}}{2\tau}$$

$$g'_\delta(0) = \lim_{\tau\to 0} \left(\frac{-v\dfrac{g'(\tau)}{g(\tau)} + v\dfrac{g'(\tau)}{g(\tau)} + \tau\dfrac{g'(\tau)}{g(\tau)} + \tau\dfrac{g'(\tau)}{g(\tau)} + \tau(v+\tau)\dfrac{g''(\tau)g(\tau)-(g'(\tau))^2}{(g(\tau))^2}}{2\tau} \right)$$

$$g'_\delta(0) = \lim_{\tau\to 0} \left(\frac{g'(\tau)}{g(\tau)} + \frac{1}{2}(v+\tau)\frac{g''(\tau)g(\tau)-(g'(\tau))^2}{(g(\tau))^2} \right)$$

$$g'_\delta(0) = \frac{g'(0)}{g(0)} + \frac{1}{2}\frac{vg''(0)g(0)-(g'(0))^2}{(g(0))^2}$$

$$g'_\delta(0) = -\delta \ln x_1 - (1-\delta)\ln x_2 + \frac{v\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2 \qquad (3.48)$$

Thirdly, $h_\delta(0) = \lim_{\tau\to 0} \dfrac{x_1^{-\tau} - x_2^{-\tau}}{\tau}$

$$h_\delta(0) = \lim_{\tau\to 0} \frac{-\ln x_1 x_1^{-\tau} + \ln x_2 x_2^{-\tau}}{1}$$

$$h_\delta(0) = -\ln x_1 + \ln x_2 \qquad (3.49)$$

$$h'_\delta(0) = \lim_{\tau\to 0} \frac{-\tau(\ln x_1 x_1^{-\tau} - \ln x_2 x_2^{-\tau}) - x_1^{-\tau} + x_2^{-\tau}}{\tau^2}$$

$$h'_\delta(0) = \lim_{\tau\to 0} \left(\frac{-(\ln x_1 x_1^{-\tau} - \ln x_2 x_2^{-\tau}) + \tau((\ln x_1)^2 x_1^{-\tau} - (\ln x_2)^2 x_2^{-\tau})}{2\tau} + \frac{\ln x_1 x_1^{-\tau} - \ln x_2 x_2^{-\tau}}{2\tau} \right)$$

$$h'_\delta(0) = \lim_{\tau\to 0} \frac{1}{2}\left((\ln x_1)^2 x_1^{-\tau} - (\ln x_2)^2 x_2^{-\tau} \right)$$

$$h'_\delta(0) = \frac{1}{2}\left((\ln x_1)^2 - (\ln x_2)^2 \right) \qquad (3.50)$$

So that we can obtain the limit of $f_\delta(\tau)$ and $f'_\delta(\tau)$ for $\tau \to 0$ by

$$f_\delta(0) = \lim_{\tau\to 0} f_\delta(\tau)$$

$$f_\delta(0) = \lim_{\tau\to 0} \left(h_\delta(\tau)\exp(-g_\delta(\tau)) \right)$$

$$f_\delta(0)=\lim_{\tau\to0}h_\delta(\tau)\lim_{\tau\to0}\exp(-g_\delta(\tau))$$

$$f_\delta(0)=\lim_{\tau\to0}h_\delta(\tau)\exp(-\lim_{\tau\to0}g_\delta(\tau))$$

$$f_\delta(0)=h_\delta(0)\exp(-g_\delta(0))$$

$$f_\delta(0)=(-\ln x_1+\ln x_2)\exp(v\delta\ln x_1+v(1-\delta)\ln x_2)$$

$$f_\delta(0)=(-\ln x_1+\ln x_2)x_1^{v\delta}x_2^{v(1-\delta)} \tag{3.51}$$

Also, $$f'_\delta(0)=\lim_{\tau\to0}f'_\delta(\tau)$$

$$f'_\delta(0)=\lim_{\tau\to0}\left(h'_\delta(\tau)\exp(-g_\delta(\tau))-h_\delta(\tau)\exp(-g_\delta(\tau))g'_\delta(\tau)\right)$$

$$f'_\delta(0)=\lim_{\tau\to0}h'_\delta(\tau)\lim_{\tau\to0}\exp(-g_\delta(\tau))-\lim_{\tau\to0}h_\delta(\tau)\lim_{\tau\to0}\exp(-g_\delta(\tau))\lim_{\tau\to0}g'_\delta(\tau)$$

$$f'_\delta(0)=\lim_{\tau\to0}h'_\delta(\tau)\exp(-\lim_{\tau\to0}g_\delta(\tau)-\lim_{\tau\to0}h_\delta(\tau)\exp(-\lim_{\tau\to0}g_\delta(\tau))\lim_{\tau\to0}g'_\delta(\tau)$$

$$f'_\delta(0)=h'_\delta(0)\exp(-g_\delta(0))-h_\delta(0)\exp(-g_\delta(0))g'_\delta(0)$$

$$f'_\delta(0)=\exp(-g_\delta(0))(h'_\delta(0)-h_\delta(0)g'_\delta(0))$$

$$f'_\delta(0)=\exp\left(v\delta\ln x_1+v(1-\delta)\ln x_2\right)$$
$$\left(\frac{1}{2}\left(\left(\ln x_1\right)^2-\left(\ln x_2\right)^2\right)-\left(-\ln x_1+\ln x_2\right)\left(-\delta\ln x_1-(1-\delta)\ln x_2+\frac{v\delta(1-\delta)}{2}\left(\ln x_1-\ln x_2\right)^2\right)\right)$$

$$f'_\delta(0)=x_1^{v\delta}+x_2^{v(1-\delta)}\left(\frac{1}{2}\left(\ln x_1\right)^2-\frac{1}{2}\left(\ln x_2\right)^2+\ln x_1-\ln x_2\left(-\delta\ln x_1-(1-\delta)\ln x_2+\frac{v\delta(1-\delta)}{2}\left(\ln x_1-\ln x_2\right)^2\right)\right)$$

$$f'_\delta(0)=x_1^{v\delta}+x_2^{v(1-\delta)}(\frac{1}{2}(\ln x_1)^2-\frac{1}{2}(\ln x_2)^2-\delta(\ln x_1)^2-(1-\delta)\ln x_1\ln x_2+\frac{v\delta(1-\delta)}{2}\ln x_1(\ln x_1-\ln x_2)^2+\delta\ln x_1\ln x_2$$
$$+(1-\delta)(\ln x_2)^2-\frac{v\delta(1-\delta)}{2}\ln x_2(\ln x_1-\ln x_2)^2)$$

$$f'_\delta(0)=x_1^{v\delta}x_2^{v(1-\delta)}((\frac{1}{2}-\delta)(\ln x_1)^2+(\frac{1}{2}-\delta)(\ln x_2)^2-2(\frac{1}{2}-\delta)\ln x_1\ln x_2$$
$$+\frac{v\delta(1-\delta)}{2}(\ln x_1-\ln x_2)(\ln x_1-\ln x_2)^2)$$

$$f'_\delta(0)=x_1^{v\delta}x_2^{v(1-\delta)}\left(\left(\frac{1}{2}-\delta\right)(\ln x_1-\ln x_2)^2+\frac{v\delta(1-\delta)}{2}(\ln x_1-\ln x_2)(\ln x_1-\ln x_2)^2\right)$$

$$f_\delta'(0) = x_1^{v\delta} x_2^{v(1-\delta)} \left(\frac{1}{2} - \delta + \frac{v\delta(1-\delta)}{2} (\ln x_1 - \ln x_2) \right) (\ln x_1 - \ln x_2)^2$$

$$f_\delta'(0) = \frac{1 - 2\delta + v\delta(1-\delta)(\ln x_1 - \ln x_2)}{2} x_1^{v\delta} x_2^{v(1-\delta)} (\ln x_1 - \ln x_2)^2 \tag{3.52}$$

Substitute equation (3.51) and (3.52) into equation (3.40) approximate $\frac{dy}{d\delta}$ by

$$\frac{dy}{d\delta} \approx -\gamma v \left(f_\delta(0) + \tau f_\delta'(0) \right) e^u$$

$$\frac{dy}{d\delta} \approx -\gamma v \left(\left(-\ln x_1 + \ln x_2 \right) x_1^{v\delta} x_2^{v(1-\delta)} + \tau \frac{1 - 2\delta + v\delta(1-\delta)(\ln x_1 - \ln x_2)}{2} x_1^{v\delta} x_2^{v\delta(1-\delta)} (\ln x_1 - \ln x_2)^2 \right) e^u$$

$$\frac{dy}{d\delta} \approx \gamma v \left(\left(\ln x_1 - \ln x_2 \right) x_1^{v\delta} x_2^{v(1-\delta)} - \tau \frac{1 - 2\delta + v\delta(1-\delta)(\ln x_1 - \ln x_2)}{2} x_1^{v\delta} x_2^{v\delta(1-\delta)} (\ln x_1 - \ln x_2)^2 \right) e^u$$

$$\frac{dy}{d\delta} \approx \gamma v \left(\ln x_1 - \ln x_2 \right) x_1^{v\delta} x_2^{v(1-\delta)} \left(1 - \tau \frac{1 - 2\delta + v\delta(1-\delta)}{2} (\ln x_1 - \ln x_2) \right) e^u \tag{3.53}$$

Derivative with respect to “ *V*”

$$\frac{dy}{dv} \approx -\gamma \frac{1}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau})(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau})^{-\frac{v}{\tau}} e^u$$

Let $\quad f_v(\tau) = \frac{1}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau})(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau})^{-\frac{v}{\tau}}$

$$f_v(\tau) = \frac{1}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau}) \exp\left(-\frac{v}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau}) \right)$$

Using Taylor series to approximate $f_v(\tau)$

$$\frac{dy}{dv} = -\gamma f_v(\tau) e^u$$

$$\frac{dy}{dv} \approx -\gamma (f_v(0) + \tau f_v'(0)) e^u \tag{3.54}$$

Derive the helper function, $g_v(\tau)$

$$g_v(\tau) = \frac{1}{\tau} \ln(\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau})$$

$$g_v(\tau) = \frac{1}{\tau}\ln(g(\tau)) \tag{3.55}$$

The first and second derivative

$$g_v'(\tau) = \frac{\tau\frac{g'(\tau)}{g(\tau)} - \ln(g(\tau))}{\tau^2}$$

$$g_v'(\tau) = \frac{1}{\tau}\frac{g'(\tau)}{g(\tau)} - \frac{\ln(g(\tau)}{\tau^2}) \tag{3.56}$$

$$g_v''(\tau) = -\frac{1}{\tau^2}\frac{g'(\tau)}{g(\tau)} + \frac{g''(\tau)}{g(\tau)} - \frac{1}{\tau}\frac{(g'(\tau))^2}{(g(\tau))^2} + 2\frac{\ln(g(\tau))}{\tau^3} - \frac{1}{\tau^2}\frac{g'(\tau)}{g(\tau)}$$

$$g_v''(\tau) = \frac{-2\frac{\tau g'(\tau)}{g(\tau)} + \frac{\tau^2 g''(\tau)}{g(\tau)} - \frac{\tau^2(g'(\tau))^2}{(g(\tau))^2} + 2\ln(g(\tau))}{\tau^3} \tag{3.57}$$

Using the function $f(\tau)$ defined above, so that

$$f_v(\tau) = g_v(\tau)\exp(f(\tau)) \tag{3.58}$$

And $$f_v'(\tau) = g_v'(\tau)\exp(f(\tau) + g_v(\tau)\exp(f(\tau))f'(\tau)) \tag{3.59}$$

Calculating the limits of equation (3.75), (3.56) and (3.57) for $\tau \to 0$ by

$$g_v(0) = \lim_{\tau\to 0} g_v(\tau)$$

$$g_v(0) = \lim_{\tau\to 0}\frac{\ln g_v(\tau)}{\tau}$$

$$g_v(0) = \lim_{\tau\to 0}\frac{\frac{g_v'(\tau)}{g(\tau)}}{1}$$

$$g_v(0) = -\delta\ln x_1 - (1-\delta\ln x_2) \tag{3.60}$$

$$g_v'(0) = \lim_{\tau\to 0} g_v'(\tau)$$

$$g_v'(0) = \lim_{\tau\to 0}\left(\frac{1}{\tau}\frac{g'(\tau)}{g(\tau)} - \frac{\ln(g(\tau))}{\tau^2}\right)$$

$$g_v'(0) = \lim_{\tau \to 0} \frac{\tau \dfrac{g'(\tau)}{g(\tau)} - \ln(g(\tau))}{\tau^2}$$

$$g_v'(0) = \lim_{\tau \to 0} \frac{\dfrac{g'(\tau)}{g(\tau)} + \tau \dfrac{g''(\tau)g(\tau) - (g'(\tau))^2}{(g(\tau))^2} - \dfrac{g'(\tau)}{g(\tau)}}{2\tau}$$

$$g_v'(0) = \lim_{\tau \to 0} \frac{g''(\tau)g(\tau) - (g'(\tau))^2}{2(g(\tau))^2}$$

$$g_v'(0) = \frac{g''(0)g(0) - (g'(0))^2}{2(g(0))^2}$$

$$g_v'(0) = \frac{1}{2}\left(\delta(\ln x_1)^2 + (1-\delta)(\ln x_2)^2 - (-\delta \ln x_1 - (1-\delta)\ln x_2)^2\right)$$

$$g_v'(0) = \frac{1}{2}\left(\delta(\ln x_1)^2 + (1-\delta)(\ln x_2)^2 - \delta^2(\ln x_1)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2 - (1-\delta)^2(\ln x_2)^2\right)$$

$$g_v'(0) = \frac{1}{2}\left((\delta - \delta^2)(\ln x_1)^2 + ((1-\delta) - (1-\delta)^2)(\ln x_2)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2\right)$$

$$g_v'(0) = \frac{1}{2}\left(\delta(1-\delta)(\ln x_1)^2 + (1-\delta)(1-(1-\delta))(\ln x_2)^2 - 2\delta(1-\delta)\ln x_1 \ln x_2\right)$$

$$g_v'(0) = \frac{\delta(1-\delta)}{2}\left((\ln x_1)^2 - 2\ln x_1 \ln x_2 + (\ln x_2)^2\right)$$

$$g_v'(0) = \frac{\delta(1-\delta)}{2}\left(\ln x_1 - \ln x_2\right)^2 \tag{3.61}$$

We can calculate the limit of $f_v(\tau)$ and $f_v'(\tau)$ for $\tau \to 0$ by

$$f_v(0) = \lim_{\tau \to 0} f_v(\tau)$$

$$f_v(0) = \lim_{\tau \to 0}(g_v(\tau)\exp(f(\tau)))$$

$$f_v(0) = \lim_{\tau \to 0} g_v(\tau) \lim_{\tau \to 0} \exp(f(\tau))$$

$$f_v(0) = \lim_{\tau \to 0} g_v(\tau) \exp\left(\lim_{\tau \to 0} f(\tau)\right)$$

$$f_v(0) = g_v(0)\exp\left(f(0)\right)$$

$$f_v(0) = \left(-\delta \ln x_1 - (1-\delta)\ln x_2\right)\exp\left(v(\delta \ln x_1 + (1-\delta)\ln x_2)\right)$$

$$f_v(0) = -\left(\delta \ln x_1 + (1-\delta)\ln x_2\right) x_1^{v\delta} x_2^{v(1-\delta)} \qquad (3.62)$$

$$f_v'(0) = \lim_{\tau \to 0} f_v'(\tau)$$

$$f_v'(0) = \lim_{\tau \to 0}\left(g_v'(\tau)\exp(f(\tau)) + g_v(\tau)\exp(f(\tau))f'(\tau)\right)$$

$$f_v'(0) = \lim_{\tau \to 0} g_v'(\tau)\exp\left(\lim_{\tau \to 0} f(\tau)\right) + \lim_{\tau \to 0} g_v(\tau)\exp\left(\lim_{\tau \to 0} f(\tau)\right)\lim_{\tau \to 0} f'(\tau)$$

$$f_v'(0) = g_v'(0)\exp(f(0)) + g_v(0)\exp(f(0))f'(0)$$

$$f_v'(0) = \exp(f(0))(g_v'(0) + g_v(0)f'(0))$$

$$f_v'(0) = \exp\left(v(\delta \ln x_1 + (1-\delta)\ln x_2)\right)\begin{pmatrix} \frac{\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2 + (-\delta \ln x_1 - (1-\delta)\ln x_2) \\ \left(-\frac{v\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2\right) \end{pmatrix}$$

$$f_v'(0) = x_1^{v\delta} x_2^{v(1-\delta)} \frac{\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2(1 + v(\delta \ln x_1 + (1-\delta)\ln x_2))$$

And approximate $\frac{\partial y}{\partial v}$ by

$$\frac{\partial y}{\partial v} \approx -\gamma(f_v(0) + \tau f_v'(0))e^u$$

$$\frac{\partial y}{\partial v} \approx \gamma e^u(\delta \ln x_1 + (1-\delta)\ln x_2)x_1^{v\delta} x_2^{v\delta(1-\delta)} - \gamma e^u \tau x_1^{v\delta} x_2^{v(1-\delta)} \frac{\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2(1 + v(\delta \ln x_1 + (1-\delta)\ln x_2))$$

$$\frac{\partial y}{\partial v} \approx \gamma e^u x_1^{v\delta} x_2^{v(1-\delta)}\left(\delta \ln x_1 + (1-\delta)\ln x_2) - \frac{\tau\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2(1 + v(\delta \ln x_1 + (1-\delta)\ln x_2))\right) \qquad (3.63)$$

Derivatives with respect to " τ"

$$\frac{\partial y}{\partial \tau} = \gamma e^{u} \frac{v}{\tau^2}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}} \ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)$$
$$+\gamma e^{u} \frac{v}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\left(\frac{v}{\tau}+1\right)}\left(\delta x_1^{-\tau} \ln x_1 + (1-\delta)x_2^{-\rho} \ln x_2\right)$$

$$\begin{aligned}\frac{\partial y}{\partial \tau} &= \gamma e^{u} v\Big(\frac{1}{\tau^2}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}} \ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right) \\ &\quad + \frac{1}{\tau}(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau})^{-\left(\frac{v}{\tau}+1\right)}\left(\delta x_1^{-\tau} \ln x_1 + (1-\delta)x_2^{-\tau} \ln x_2\right)\Big)\end{aligned} \tag{3.64}$$

Now, defining the function $f_\tau(\tau)$

$$\begin{aligned}f_\tau(\tau) &= \frac{1}{\tau^2}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}} \ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right) \\ &\quad + \frac{1}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\left(\frac{v}{\tau}+1\right)}\left(\delta x_1^{-\tau} \ln x_1 + (1-\delta)x_2^{-\tau} \ln x_2\right)\end{aligned} \tag{3.65}$$

So that $\frac{\partial y}{\partial \tau}$ can be approximated using the first-order Taylor series approximation of $f_\tau(\tau)$:

$$\frac{\partial y}{\partial \tau} = \gamma v f_\tau(\tau) e^{u}$$

$$\frac{\partial y}{\partial \tau} \approx \gamma v(f_\tau(0) + \tau f_\tau'(0))e^{u} \tag{3.66}$$

Define the help function $g_\tau(\tau)$

$$g_\tau(\tau) = \delta x_1^{-\tau} \ln x_1 + (1-\delta)x_2^{-\tau} \ln x_2 \tag{3.67}$$

The first and second derivative

$$g_\tau'(\tau) = -\delta x_1^{-\tau}(\ln x_1)^2 - (1-\delta)x_2^{-\tau}(\ln x_2)^2 \tag{3.68}$$

$$g_\tau''(\tau) = \delta x_1^{-\tau}(\ln x_1)^3 - (1-\delta)x_2^{-\tau}(\ln x_2)^3 \tag{3.69}$$

Then use the functions $g(\tau)$ and $g_v(\tau)$ all defined above so that

$$f_\tau(\tau) = \frac{1}{\tau^2}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}} \ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)$$
$$+\frac{1}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\left(\frac{v}{\tau}+1\right)}\left(\delta x_1^{-\tau}\ln x_1 + (1-\delta)x_2^{-\tau}\ln x_2\right)$$

$$f_\tau(\tau) = \frac{1}{\tau^2}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}} \ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)$$
$$+\frac{1}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-1}\left(\delta x_1^{-\tau}\ln x_1 + (1-\delta)x_2^{-\tau}\ln x_2\right)$$

$$f_\tau(\tau) = \frac{1}{\tau}\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-\frac{v}{\tau}}\left(\frac{1}{\tau}\ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right) + \left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)^{-1}\left(\delta x_1^{-\tau}\ln x_1 + (1-\delta)x_2^{-\tau}\ln x_2\right)\right)$$

$$f_\tau(\tau) = \frac{1}{\tau}\exp\left(-\frac{v}{\tau}\ln\left(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right)\right)$$
$$\left(\frac{1}{\tau}\ln(\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}) + (\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau})^{-1}(\delta x_1^{-\tau}\ln x_1 + (1-\delta)x_2^{-\tau}\ln x_2)\right)$$

$$f_\tau(\tau) = \frac{\exp(-vg_v(\tau))\left(g_v(\tau) + g(\tau)^{-1}g_\tau(\tau)\right)}{\tau} \tag{3.70}$$

And we can calculate its first derivative below as;

$$f_\tau(\tau) = \frac{-\tau v\exp(-vg_v(\tau))g_v'(\tau)\left(g_v(\tau) + g(\tau)^{-1}g_\tau(\tau)\right)}{\tau^2}$$
$$+\frac{\tau\exp(-vg_v(\tau))\left(g_v'(\tau) - g(\tau)^{-2}g'(\tau)g_\rho(\tau) + g(\tau)^{-1}g_\tau'(\tau)\right)}{\tau^2}$$
$$-\frac{\exp(-vg_v(\tau))\left(g_v(\tau) + g(\tau)^{-1}g_\tau(\tau)\right)}{\tau^2}$$

$$f_\tau(\tau) = \frac{\exp(-vg_v(\tau))\tau\left(-vg_v'(\tau)g_v(\tau) - vg_v'(\tau)g(\tau)^{-1}g_\tau(\tau)\right)}{\tau^2}$$
$$+\frac{\exp(-vg_v(\tau))\tau\left(g_v'(\tau) - g(\tau)^{-2}g'(\tau)g_\tau(\tau) + g(\tau)^{-1}g_\tau'(\tau)\right)}{\tau^2}$$
$$-\frac{\exp(-vg_v(\tau))\left(g_v(\tau) + g(\tau)^{-1}g_\tau(\tau)\right)}{\tau^2}$$

$$f_\tau(\tau)=\frac{\exp(-vg_v(\tau))}{\tau^2}(\tau(-vg_v'(\tau)g_v(\tau)-vg_v'(\tau)g(\tau)^{-1}g_\tau(\tau)$$
$$+g_v'(\tau)-g(\tau)^{-2}g'(\tau)g(\tau)+g(\tau)^{-1}g_\tau'(\tau))-g_v(\tau)-g(\tau)^{-1}g_\tau(\tau))$$

$$f_\tau(\tau)=\frac{\exp(-vg_v(\tau))}{\tau^2}((-v\tau g_v'(\tau)g_v(\tau)-v\tau g_v'(\tau)g(\tau)^{-1}g_\tau(\tau)$$
$$+\tau g_v'(\tau)-\tau g(\tau)^{-2}g'(\tau)g(\tau)+\tau g(\tau)^{-1}g_\tau'(\tau))-g_v(\tau)-g(\tau)^{-1}g_\tau(\tau)) \tag{3.71}$$

Calculate the limits of $g_\tau(\tau)$ and $g_\tau'(\tau)$ for $\tau\to 0$ by

$$g_\tau(0)=\lim_{\tau\to 0} g_\tau(\tau)$$

$$g_\tau(0)=\lim_{\tau\to 0}\left(\delta x_1^{-\tau}\ln x_1+(1-\delta)x_2^{-\tau}\ln x_2\right)$$

$$g_\tau(0)=\delta\ln x_1\lim_{\tau\to 0}x_1^{-\tau}+(1-\delta)\ln x_2\lim_{\tau\to 0}x_2^{-\tau}$$

$$g_\tau(0)=\delta\ln x_1+(1-\delta)\ln x_2 \tag{3.72}$$

$$g_\tau'(0)=\lim_{\tau\to 0} g_\tau'(\tau)$$

$$g_\tau'(0)=\lim_{\tau\to 0}\left(-\delta x_1^{-\tau}(\ln x_1)^2-(1-\delta)x_2^{-\tau}(\ln x_2)^2\right)$$

$$g_\tau'(0)=-\delta(\ln x_1)^2\lim_{\tau\to 0}x_1^{-\tau}-(1-\delta)(\ln x_2)^2\lim_{\tau\to 0}x_2^{-\tau}$$

$$g_\tau'(0)=-\delta(\ln x_1)^2-(1-\delta)(\ln x_2)^2 \tag{3.73}$$

Therefore, the limit of $f_\tau(\tau)$ for $\tau\to 0$ by

$$f_\tau(0)=\lim_{\tau\to 0} f_\tau(\tau)$$

$$f_\tau(0)=\lim_{\tau\to 0}\left(\frac{\exp(-vg_v(\tau))(g_v(\tau)+g(\tau)^{-1}g_\tau(\tau))}{\tau}\right)$$

$$f_\tau(0)=\lim_{\tau\to 0}(\frac{-v\exp(-vg_v(\tau))g_v'(\tau)((g_v(\tau)+g(\tau)^{-1}g_\tau(\tau))}{1}$$
$$+\frac{\exp(-vg_v(\tau))(g_v'(\tau)-g(\tau)^{-2}g'(\tau)g_\tau(\tau)+g(\tau)^{-1}g_\tau'(\tau))}{1})$$

$$f_\tau(0) = -v\exp(-vg_v(0))g_v'(0)\left(g_v(0) + g(0)^{-1}g_\tau(0)\right)$$
$$+\exp(-vg_v(0))\left(g_v'(0) - g(0)^{-2}g'(0)g_\tau(0) + g(0)^{-1}g_\tau'(0)\right)$$

$$f_\tau(0) = \exp(-vg_v(0))(-vg_v'(0))\left(g_v(0) + g(0)^{-1}g_\tau(0)\right)$$
$$+\exp(-vg_v(0))\left(g_v'(0) - g(0)^{-2}g'(0)g_\tau(0) + g(0)^{-1}g_\tau'(0)\right)$$

$$f_\tau(0) = \exp(-vg_v(0))\Big((-vg_v'(0))\left(g_v(0) + g(0)^{-1}g_\tau(0)\right)$$
$$+g_v'(0) - g(0)^{-2}g'(0)g_\tau(0) + g(0)^{-1}g_\tau'(0)\Big)$$

$$f_\tau(0) = \exp(-v(-\delta\ln x_1 - (1-\delta)\ln x_2))\Big(-\frac{v\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2(-\delta\ln x_1 - (1-\delta)\ln x_2 + \delta\ln x_1 + (1-\delta)\ln x_2)$$
$$+\frac{\delta(1-\delta)}{2}(\ln x_1 - \ln x_2)^2 - (-\delta\ln x_1 - (1-\delta)\ln x_2)(\delta\ln x_1 + (1-\delta)\ln x_2) - \delta(\ln x_1)^2 - (1-\delta)(\ln x_2)^2\Big)$$

$$f_\tau(0) = x_1^{v\delta}x_2^{v(1-\delta)}\Big(\frac{1}{2}\delta(1-\delta)(\ln x_1)^2 - \delta(1-\delta)\ln x_1\ln x_2 + \frac{1}{2}\delta(1-\delta)(\ln x_2)^2 + \delta^2(\ln x_1)^2 + 2\delta(1-\delta)\ln x_1\ln x_2$$
$$+(1-\delta)^2(\ln x_2)^2 - \delta(\ln x_1)^2 - (1-\delta)(\ln x_2)^2\Big)$$

$$f_\tau(0) = x_1^{v\delta}x_2^{v(1-\delta)}\left(\left(\frac{1}{2}\delta(1-\delta) + \delta^2 - \delta\right)(\ln x_1)^2 + \left(\frac{1}{2}\delta(1-\delta) + (1-\delta)^2 - (1-\delta)\right)\right.$$
$$\left.(\ln x_2)^2 + \delta(1-\delta)\ln x_1\ln x_2\right)$$

$$f_\tau(0) = x_1^{v\delta}x_2^{v(1-\delta)}\left(\left(\frac{1}{2}\delta - \frac{1}{2}\delta^2 + \delta^2 - \delta\right)(\ln x_1)^2 + \left(\frac{1}{2}\delta - \frac{1}{2}\delta^2 + 1 - 2\delta + \delta^2 - 1 + \delta)\right)\right.$$
$$\left.(\ln x_2)^2 + \delta(1-\delta)\ln x_1\ln x_2\right)$$

$$f_\tau(0) = x_1^{v\delta}x_2^{v(1-\delta)}\left(\left(-\frac{1}{2}\delta + \frac{1}{2}\delta^2\right)(\ln x_1)^2 + \left(-\frac{1}{2}\delta + \frac{1}{2}\delta^2\right)(\ln x_2)^2 + \delta(1-\delta)\ln x_1\ln x_2\right)$$

$$f_\tau(0) = x_1^{v\delta}x_2^{v(1-\delta)}\left(-\frac{1}{2}\delta(1-\delta)(\ln x_1)^2 - \frac{1}{2}\delta(1-\delta)(\ln x_2)^2 + \delta(1-\delta)\ln x_1\ln x_2\right)$$

$$f_\tau(0) = -\frac{1}{2}\delta(1-\delta)x_1^{v\delta}x_2^{v(1-\delta)}\left((\ln x_1)^2 + (\ln x_2)^2 - 2\ln x_1\ln x_2\right)$$

$$f_\tau(0) = -\frac{1}{2}\delta(1-\delta)x_1^{v\delta}x_2^{v(1-\delta)}\left(\ln x_1 - \ln x_2\right)^2 \qquad (3.74)$$

Substituting where necessary in the Taylor series approximation in equation (3.35) we obtain the Kmenta approximation below;

$$\ln y = \varphi_0 + \varphi_1 \ln x_1 + \varphi_2 \ln x_2 + \frac{1}{2} B_{11} (\ln x_1)^2 + \frac{1}{2} B_{22} (\ln x_2)^2 + B_{12} \ln x_1 \ln x_2 \quad (3.75)$$

Where; $\gamma = \exp(\varphi_0)$, $v = \varphi_1 + \varphi_2$, $\delta = \frac{\varphi_1}{\varphi_1 + \varphi_2}$, $\tau = \frac{B_{12}(\varphi_1 + \varphi_2)}{\varphi_1 \varphi_2}$, and $B_{12} = -B_{11} = -B_{22}$

3.2 Bayesian Estimators of Nonlinear Production Functions

A Bayesian data analysis using Markov Chain Monte Carlo technique (MCMC) often involves a confusing array of sampling procedures and probability distribution (Hoff 2009). With this awareness, one can distinguish the statistical data analysis from that of numerical approximation.

3.2.1 The Non-Linear Production functions with Independent Normal-Gamma Prior

The Cobb-Douglas Model with Multiplicative Error Term.

Given the following production function

$$y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} e^{u_i} \quad (3.76)$$

Linearizing equation (3.76) by taking natural log of both sides of the equation to obtain:

$$\ln(y_i) = \ln(\beta_0) + \beta_1 \ln(X_{i1}) + \beta_2 \ln(X_{i2}) + u_i \quad (3.77)$$

The specification in equation (3.77) is now linear in logs of the dependent and the explanatory variables entered in an intrinsically linear way can further be expressed as

$$Y^* = \beta^*_0 + \beta_1 X^*_1 + \beta_2 X^*_2 + \varepsilon_i \quad (3.78)$$

where

$\ln(y_i) = Y^*$

$\ln(\beta_0) = \beta_0^*$

$\ln(X_{i1}) = X_1^*$

$\ln(X_{i2}) = X_2^*$

$u_i = \varepsilon_i \sim N(0, \sigma^2 I)$

3.2.2 The Likelihood Function

If the log normal regression mean $X^*\beta^*$ and error precision is $h = \frac{1}{\sigma^2}$, with random variable $\ln(y^*)$ as the data information, the expression for the likelihood density can be generally written as:

$$L^*(\theta) = \sum_{i=1}^{N} \ln[f(y_i^* \mid \theta)] \quad (3.79)$$

where, $L^*(\theta)$ is the likelihood function for the transformed model, for a normally distributed variable of interest we can write the log likelihood as follows:

$$L^*(\theta) = \sum_{i=1}^{N} [\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(y^* - X^*\beta)'(y^* - X^*\beta))] \quad (3.80)$$

Concretely, using the multivariate normal density to denote the log likelihood for the above observation as follows:

$$P\left(y^* \mid \beta^*, h^*\right)$$

which is given as:

$$P\left(y^* \mid \beta^*, h^*\right) = \frac{h^{*\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} \exp[-\frac{h^*}{2}(y^* - X^*\beta^*)'(y^* - X^*\beta^*)] \quad (3.81)$$

3.2.3 The Prior

The non informative prior used in this study is independent normal –gamma denoted by $P\left(\beta^*, h^*\right)$. This type of prior is used because of the little or no information available about the variable. In the independent random variables, it follows that,

$$P\left(\beta^*, h^*\right) = P\left(\beta^*\right) \cdot P\left(h^*\right)$$

$P\left(\beta^*\right)$isnormal and $P\left(h^*\right)$ is Gamma:

$$P\left(\beta^*\right) = \frac{1}{\left(2\pi\right)^{\frac{N}{2}}} \left|\underline{V}\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\beta^* - \underline{\beta^*}\right)' \underline{V}^{-1}\left(\beta^* - \underline{\beta^*}\right)\right]$$

and

$$P\left(h^*\right) = C_G^{-1} h^{*\frac{\underline{v}-2}{2}} \exp\left(\frac{-h^* \underline{v}}{2\underline{s}^{-2}}\right)$$

where, C_G is the integrating constant for Gamma, it is deduced that: $E\left[\beta^* \mid y^*\right] = \underline{\beta}^*$ is the prior mean of β^* and $Var\left(\beta^* \mid h^*\right) = \underline{V^*}$ is the prior covariance matrix of β^* with which the mean of h^* is $\underline{s}^{-2}$and $\underline{v}^*$ degree of freedom.

3.2.4 The Posterior Density

The posterior density is proportional to the product of prior and the likelihood. This implies that information obtained after seeing the data and some mathematical techniques applied could be conjugate or independent or having numerically imposible and unfamiliar distribution form denoted by $P\left(\beta^*, h^* \mid y^*\right)$.

Note:

$$P\left(\beta^*, h^* \mid y^*\right) \neq P\left(\beta^* \mid y^*, h^*\right) \cdot P\left(h^* \mid y^*, \beta^*\right)$$

Then, the posterior:

$$P\left(\beta^*, h^* \mid y^*\right) \to L(Y^* \mid \beta^*, h^*).p(\beta^*, h^*)$$

$$P\left(\beta^*, h^* \mid y^*\right) = \frac{h^{*\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} \exp\left[-\frac{h^*}{2}\left(Y^* - X^*\beta^*\right)'\left(Y^* - X^*\beta^*\right)\right] . \frac{1}{(2\pi)^{\frac{N}{2}}} |V|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\beta^* - \underline{\beta}^*\right)' \underline{V}^{-1}\left(\beta^* - \underline{\beta}^*\right)\right] \cdot$$

$$C_G^{-1} h^{*\frac{\underline{v}-2}{2}} \exp\left(\frac{-h^* \underline{v}}{2\underline{s}^{-2}}\right)$$

$$P\left(\beta^*, h^* \mid y^*\right) \propto \exp\left[-\frac{1}{2}\{h^*\left(Y^* - X^*\beta^*\right)'\left(Y^* - X^*\beta^*\right) + \left(\beta^* - \underline{\beta}^*\right)' \underline{V}^{-1}\left(\beta^* - \underline{\beta}^*\right)\}\right] \cdot h^{*\frac{N+\underline{v}-2}{2}} \exp\left(\frac{-h^* \underline{v}}{2\underline{s}^{-2}}\right) \quad (3.82)$$

this joint posterior density for β^* and h^* does not take any well-known distributional form; so it cannot be solved analytically but only through a posterior simulation method.

The Matrix Multiplication to Simplify the Posterior Distribution of Cobb-Douglas (non-linear) regression Model with multiplication Error Term.

From equation (3.82), the expression in the outermost parentheses ignoring ($-\frac{1}{2}$) becomes;

$$h^*\left(Y^* - X^*\beta^*\right)'(Y^* - X^*\beta^*) + (\beta^* - \underline{\beta}^*)'\underline{V}^{-1}(\beta^* - \underline{\beta}^*)$$
$$= h^*\left(Y^{*\prime}Y^* - 2\beta^{*\prime}X^{*\prime}Y + \beta^* X^{*\prime}X^*\beta^*\right) + (\beta^* - \underline{\beta}^*)'\underline{V}^{-1}(\beta^* - \underline{\beta}^*)$$

$$= h^*Y^{*\prime}Y^* - 2h^*\beta' X^{*\prime}Y^* + h^*\beta^{*\prime}X^{*\prime}X^*\beta^* + \beta^{*\prime}\underline{V}^{-1}\beta^* - 2\beta^{*\prime}\underline{V}^{-1}\underline{\beta} + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^*$$

$$= h^*Y^{*\prime}Y^* + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^{*\prime} + \beta^{*\prime}(\underline{V}^{-1} + h^*X^{*\prime}X^*)\beta^* - 2\beta^{*\prime}(h^*X^{*\prime}Y^* + \underline{V}^{-1}\underline{\beta}^*) \quad (3.83)$$

Let $\overline{V^*} = (\underline{V}^{-1} + h^*X^{*\prime}X^*)^{-1}$ (3.84)

$$\Rightarrow \overline{V}^{-1} = (\underline{V}^{-1} + h^*X^{*\prime}X^*) \quad (3.85)$$

$$\overline{\beta}^* = \overline{V^*}(h^*X^{*\prime}Y^* + \underline{V}^{-1}\underline{\beta}^*) \quad (3.86)$$

Hence, substituting back (3.84), (3.85) and (3.86) into (3.83) to have,

$$h^*Y^{*\prime}Y^* + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^* + \beta^{*\prime}\overline{V}^{-1}\beta^* - 2\beta^{*\prime}\overline{V}^{-1}\overline{\beta}^* \tag{3.87}$$

carrying out a simple mathematical assumption by including $-\overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$ and $+\overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$ into (3.87) above which does not change anything in the equation but achieve the desired result as follows

$$= h^*Y^{*\prime}Y^* + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^* + \beta^{*\prime}\overline{V}^{-1}\beta^* - 2\beta^{*\prime}\overline{V}^{-1}\overline{\beta}^* - \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^* + \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$$

$$= h^*Y^{*\prime}Y^* + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^* - \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^* + \beta^{*\prime}\overline{V}^{-1}\beta^* - 2\beta^{*\prime}\overline{V}^{-1}\overline{\beta}^* + \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$$

Let,

$$Q = h^*Y^{*\prime}Y^* + \underline{\beta}^{*\prime}\underline{V}^{-1}\underline{\beta}^* - \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$$

and

$$(\beta^* - \overline{\beta}^*)'\overline{V}^{-1}(\beta^* - \overline{\beta}^*) = \beta^{*\prime}\overline{V}^{-1}\beta^* - 2\beta^{*\prime}\overline{V}^{-1}\overline{\beta}^* + \overline{\beta}^{*\prime}\overline{V}^{-1}\overline{\beta}^*$$

$$\Rightarrow h^*(Y^* - X^*\beta^*)'(Y^* - X^*\beta^*) + (\beta^* - \underline{\beta}^*)'\underline{V}^{-1}(\beta^* - \underline{\beta}^*) = (\beta^* - \overline{\beta}^*)'\overline{V}^{-1}(\beta^* - \overline{\beta}^*) + Q \tag{3.88}$$

Substituting the expression in equation (3.82) for the expression in equation (3.88), we obtain

$$P\left(\beta^*, h^* \mid y^*\right) \propto \exp\left[-\frac{1}{2}\left\{\left(\beta^* - \overline{\beta}^*\right)'\overline{V}^{-1}\left(\beta^* - \overline{\beta}^*\right)\right\}\right] \bullet \exp[-\frac{1}{2}Q]h^{*\frac{N+\underline{v}-2}{2}}\exp\left(\frac{-h^*\underline{v}}{2\underline{s}^{-2}}\right) \tag{3.89}$$

By ignoring the terms that do not involve β^* in equation (3.89) we obtain,

$$P\left(\beta^* \mid y^*, h^*\right) \propto \exp\left[-\frac{1}{2}\left\{\left(\beta^* - \overline{\beta}^*\right)'\overline{V}^{-1}\left(\beta^* - \overline{\beta}^*\right)\right\}\right] \tag{3.90}$$

Which implies that $\beta^* \mid y^*, h^* \sim N(\overline{\beta}^*, \overline{V}^*)$, a **Multivariate Normal density**

Where, $\overline{V}^* = (\underline{V}^{-1} + h^* X^{*\prime} X^*)^{-1}$ and $\overline{\beta} = \overline{V}(hX^{*\prime}Y^* + \underline{V}^{-1}\underline{\beta})$

Similarly, by treating equation (3.89) as a function of h ignoring terms that do not involve h we can obtain

$$P\left(h^* \mid y^*, \beta^*\right) \propto h^{*\frac{N+\underline{v}-2}{2}} \exp[-\frac{h^*}{2}\{(Y^* - X^*\beta^*)'(Y^* - X^*\beta^*) + \underline{vs}^2\}] \quad (3.91)$$

This also implies that $(h^* \mid y^*, \beta^*) \sim G(\overline{s}^{-2}, \overline{v})$, a **Gamma density**

Where, $\overline{v} = N + \underline{v}$ and $\overline{s}^{-2} = \frac{(Y^* - X^*\beta^*)'(Y^* - X^*\beta^*) + \underline{vs}^2}{\overline{v}}$

The formulae of equation (3.90) and (3.91) look familiar to those of the conjugate normal-gamma priors now but do not relate directly to the posteriors of interest, since $P\left(\beta^*, h^* \mid y^*\right) \neq P\left(\beta^* \mid y^*, h^*\right) \bullet P\left(h^* \mid y^*, \beta^*\right)$. Therefore, the conditional posteriors in equation (3.90) and (3.91) do not directly tell everything about the posterior, $P\left(\beta^*, h^* \mid y^*\right)$. Nevertheless, there is a posterior simulator called the Metropolis-Within-Gibbs which makes use of the conditional posteriors like (3.90) and (3.91) to produce random draws $\beta^{*(s)}$ and $h^{*(s)}$ for $s = 1, 2, ..., S$ which can be averaged to produce estimates of the posterior properties just as the Monte Carlo integration.

3.2.5 The Cobb-Douglas Model with Additive Error Term.

The nonlinear model is of the form

$$y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} + u_i \quad (3.92)$$

By factoring out the $\beta_0 X_1^{\beta_1} X_2^{\beta_2}$ from equation (3.91) we have

$$y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} \left(1 + \frac{u_i}{\beta_0 X_1^{\beta_1} X_2^{\beta_2}}\right) \quad (3.92)$$

Let $v = \frac{u_i}{\beta_0 X_1^{\beta_1} X_2^{\beta_2}}$ then equation (3.92) becomes

$$y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} \left(1+v\right) \tag{3.93}$$

Taking the log of both sides to linearize the above equation, we obtain;

$$\ln(y_i) = \ln(\beta_0) + \beta_1 \ln(X_{i1}) + \beta_2 \ln(X_{i2}) + \ln(1+v) \tag{3.94}$$

Let $p = 1 + v$, such that

$$\log_e(p) = \log_e(1+v) \tag{3.95}$$

Substituting equation (3.95) into (3.94) gives

$$\ln(y_i) = \ln(\beta_0) + \beta_1 \ln(X_{i1}) + \beta_2 \ln(X_{i2}) + \ln(p) \tag{3.96}$$

Which can be further expressed as:

$$Y^{**} = \beta^{**}{}_0 + \beta_1 X^{**}{}_1 + \beta_2 X^{**}{}_2 + u^* \tag{3.97}$$

Where,

$\ln(y_i) = Y^{**}$

$\ln(\beta_0) = \beta_0^{**}$

$\ln(X_{i1}) = X_1^{**}$

$\ln(X_{i2}) = X_2^{**}$

$\ln p = u^*$

which means that the same procedure for analyzing the "Cobb-Douglas with Multiplicative error term" is also applicable.

3.2.6 The Likelihood Function

The log normal regression mean is $X^{**}\beta$ with error precision as $h = \frac{1}{\sigma^2}$, and random variable $\ln(y)$ which is the data information, the generalized likelihood density can be written as:

$$L^{**}(\theta) = \sum_{i=1}^{N} \ln[f(y_i \mid \theta)] \tag{3.97}$$

Where, $L^{**}(\theta)$ is the likelihood function of the transformed model, then the expression of normal log likelihood is as follows:

$$L^{**}(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(y^{**} - X^{**}\beta)'(y^{**} - X^{**}\beta)) \tag{3.98}$$

The log likelihood multivariate Normal density denoted by:

$$P\left(y^{**} \mid \beta, h\right)$$

and can be given as:

$$P\left(y^{**} \mid \beta, h\right) = \frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} \exp[-\frac{h}{2}(y^{**} - X^{**}\beta)'(y^{**} - X^{**}\beta)] \tag{3.99}$$

3.2.7 The Prior

This is defined as the information we have about a particular study before seeing the data and the independent prior is denoted by $P(\beta, h)$

From the law of independent random variables, its follows that,

$$P(\beta, h) = P(\beta) \cdot P(h)$$

Where $P(\beta)$ is normal density and $P(h)$ is Gamma density

$$P(\beta) = \frac{1}{(2\pi)^{\frac{N}{2}}} |\underline{V}|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\beta - \underline{\beta}\right)' \underline{V}^{-1}\left(\beta - \underline{\beta}\right)\right]$$

and

$$P(h) = C_G^{-1} h^{\frac{\underline{v}-2}{2}} \exp\left(\frac{-h\underline{v}}{2\underline{s^{-2}}}\right)$$

Where, C_G is an integrating constant for Gamma, the $E[\beta \mid y]=\underline{\beta}$ is the prior mean of β, $Var(\beta \mid h)=\underline{V}$ is the prior covariance matrix of β, the mean of h is $\underline{s}^{-2}$ and $\underline{v}$ degree of freedom.

3.2.8 The Posterior

A posterior can be defined as the product of the prior and the likelihood, which is also the information obtained after seeing the data and some mathematical techniques being applied, it can be a conjugate or independent or not taking a familiar distribution form, usually denoted $P\left(\beta, h \mid y^{**}\right)$.

Then, the posterior:

$$P\left(\beta,h|y^{**}\right)=\frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\exp\left[-\frac{h}{2}\left(Y^{**}-X^{**}\beta\right)'\left(Y^{**}-X^{**}\beta\right)\right]\cdot\frac{1}{(2\pi)^{\frac{N}{2}}}\left|\underline{V}\right|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\beta-\underline{\beta}\right)'\underline{V}^{-1}\left(\beta-\underline{\beta}\right)\right]\cdot C_G^{-1}h^{\frac{\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right)$$

$$P\left(\beta,h|y^{**}\right)\propto\exp\left[-\frac{1}{2}\{h\left(Y^{**}-X^{**}\beta\right)'\left(Y^{**}-X^{**}\beta\right)+\left(\beta-\underline{\beta}\right)'\underline{V}^{-1}\left(\beta-\underline{\beta}\right)\}\right]\cdot h^{\frac{N+\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right) \qquad (3.100)$$

the joint posterior density for β and h in equation (3.100) does not take any well-known distributional form; so it cannot be solved analytically but only through a posterior simulation method.

The Matrix Multiplication to Simplify the Posterior Distribution of Cobb-Douglas Model with Additive Error Term.

The expression from the equation (3.100) can be given as;

$$h\left(Y^{**}-X^{**}\beta\right)'(Y^{**}-X^{**}\beta)+(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta})$$
$$= h\left(Y^{**\prime}Y^{**}-2\beta' X^{**\prime}Y^{**}+\beta X^{**\prime}X^{**}\beta\right)+(\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta})$$

$$= hY^{**\prime}Y^{**}-2h\beta' X^{**\prime}Y^{**}+h\beta' X^{**\prime}X^{**}\beta+\beta'\underline{V}^{-1}\beta-2\beta'\underline{V}^{-1}\underline{\beta}+\underline{\beta}'\underline{V}^{-1}\underline{\beta}$$

$$= hY^{**\prime}Y^{**}+\underline{\beta}'\underline{V}^{-1}\underline{\beta}+\beta'(\underline{V}^{-1}+hX^{**\prime}X^{**})\beta-2\beta'(hX^{**\prime}Y^{**}+\underline{V}^{-1}\underline{\beta}) \qquad (3.101)$$

Let $\overline{V}=(\underline{V}^{-1}+hX^{**\prime}X^{**})^{-1}$ (3.102)

$$\Rightarrow \overline{V}^{-1} = (\underline{V}^{-1} + hX^{**\prime}X^{**}) \qquad (3.103)$$

$$\overline{\beta} = \overline{V}(hX^{**\prime}Y^{**} + \underline{V}^{-1}\underline{\beta}) \qquad (3.104)$$

Hence, substituting (3.102), (3.103) and (3.104) back to equation (3.101) to have,

$$= hY^{**\prime}Y^{**} + \underline{\beta}'\underline{V}^{-1}\underline{\beta} + \beta'\overline{V}^{-1}\beta - 2\beta'\overline{V}^{-1}\overline{\beta} \qquad (3.105)$$

carrying out a simple mathematical assumption by including $-\overline{\beta}'\overline{V}^{-1}\overline{\beta}$ and $+\overline{\beta}'\overline{V}^{-1}\overline{\beta}$ into the equation (3.105) above which does not change anything in the equation but hel[ped to achieve the desired result and have;

$$= hY^{**\prime}Y^{**} + \underline{\beta}'\underline{V}^{-1}\underline{\beta} + \beta'\overline{V}^{-1}\beta - 2\beta'\overline{V}^{-1}\overline{\beta} - \overline{\beta}'\overline{V}^{-1}\overline{\beta} + \overline{\beta}'\overline{V}^{-1}\overline{\beta}$$

$$= hY^{**\prime}Y^{**} + \underline{\beta}'\underline{V}^{-1}\underline{\beta} - \overline{\beta}'\overline{V}^{-1}\overline{\beta} + \beta'\overline{V}^{-1}\beta - 2\beta'\overline{V}^{-1}\overline{\beta} + \overline{\beta}'\overline{V}^{-1}\overline{\beta}$$

Let,

$$Q = hY^{**\prime}Y^{**} + \underline{\beta}'\underline{V}^{-1}\underline{\beta} - \overline{\beta}'\overline{V}^{-1}\overline{\beta} \text{ and } (\beta-\overline{\beta})'\overline{V}^{-1}(\beta-\overline{\beta}) = \beta'\overline{V}^{-1}\beta - 2\beta'\overline{V}^{-1}\overline{\beta} + \overline{\beta}'\overline{V}^{-1}\overline{\beta}$$

$$\Rightarrow h(Y^{**} - X^{**}\beta)'(Y^{**} - X^{**}\beta) + (\beta-\underline{\beta})'\underline{V}^{-1}(\beta-\underline{\beta}) = (\beta-\overline{\beta})'\overline{V}^{-1}(\beta-\overline{\beta}) + Q \qquad (3.106)$$

Substituting the expression in equation (3.100) for the expression in equation (3.106), to obtain

$$P(\beta,h|y) \propto \exp\left[-\frac{1}{2}\{(\beta-\overline{\beta})'\overline{V}^{-1}(\beta-\overline{\beta})\}\right] \bullet \exp[-\frac{1}{2}Q]h^{\frac{N+\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right) \qquad (3.107)$$

By ignoring the terms that do not involve β in equation (3.99) to obtain,

$$P(\beta|y^{**},h) \propto \exp\left[-\frac{1}{2}\{(\beta-\overline{\beta})'\overline{V}^{-1}(\beta-\overline{\beta})\}\right] \qquad (3.108)$$

Which implies that $\beta|y^{**},h \sim N(\overline{\beta},\overline{V})$, a **Multivariate Normal density**

Where, $\overline{V}=(\underline{V}^{-1}+hX^{**\prime}X^{**})^{-1}$ and $\overline{\beta}=\overline{V}(hX^{**\prime}Y^{**}+\underline{V}^{-1}\underline{\beta})$

Similarly, by treating equation (3.107) as a function of h ignoring terms that do not involve h one can obtain

$$P\left(h\,|\,y^{**},\beta\right)\propto h^{\frac{N+\underline{v}-2}{2}}\exp[-\frac{h}{2}\{(Y^{**}-X^{**}\beta)'(Y^{**}-X^{**}\beta)+\underline{v}\underline{s}^{2}\}] \quad (3.109)$$

This also implies that $h\,|\,y^{**},\beta\sim G(\overline{s}^{-2},\overline{v})$, a **Gamma density**

Where, $\overline{v}=N+\underline{v}$ and $\overline{s}^{-2}=\dfrac{(Y^{**}-X^{**}\beta)'(Y^{**}-X^{**}\beta)+\underline{v}\underline{s}^{2}}{\overline{v}}$

The formulae of equation (3.108) and (3.109) look familiar to those of the conjugate normal-gamma priors now but it does not relate directly to the posterior of interest, since $P\left(\beta,h\,|\,y^{**}\right)\neq P\left(\beta\,|\,y^{**},h\right)\bullet P\left(h\,|\,y^{**},\beta\right)$. Therefore, the conditional posteriors in equation (3.108) and (3.109) do not directly tells everything about the posterior, $P\left(\beta,h\,|\,y^{**}\right)$.Nevertheless, there is a posterior simulator called the Metropolis-Within-Gibbs which makes use of the conditional posteriors like (3.108) and (3.109) to produce random draws $\beta^{(s)}$ and $h^{(s)}$ for $s=1,2,...,S$ which can be averaged to produce estimates of the posterior properties just as the Monte Carlo integration.

3.2.9 The CES (Non-linear) Production Function Model with a Multiplicative Error term.

$$y=\gamma\left[\delta x_1^{-\tau}+(1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}}e^{u} \quad (3.110)$$

Where, y is the response variable, γ,δ,τ and v are the nonlinear regression parameters, x_1 *and* x_2 are the explanatory variables. $u$is the error component (well behaved, i.e. $N(0,\sigma^2)$). Also, parameter $\gamma\in[0,\infty)$ determines the productivity, $\delta\in[0,1]$ determines the optimal distribution of the inputs, $\tau\in[-1,0)\cup(0,\infty)$

determines the elasticity of substitution, which is $\sigma = \dfrac{1}{(1+\tau)}$, and $v \in [0,\infty)$ is equal to the elasticity of scale.

The CES function can be written in the form

$$\ln y = \ln f(X,\gamma) + u \qquad (3.111)$$

INVESTIGATION OF THE THREE SPECIAL CASES OF THE CES FUNCTION

(i) When $\tau \to 0$, σ approaches 1, and the CES function turns to a Cobb-Douglas form i.e. $\sigma = 1/(1+0) = 1$, where labour and capital are perfect substitutes and easy to work with.

(ii) When $\tau \to \infty$, σ approaches 0 and the CES function turns to the Leontief production function. i.e. $\sigma = 1/(1+\infty) = 0$, where labour and capital are extreme complements. Leontief's theorem requires that labour has no effect on the substitution possibilities between the capital inputs.

(iii) When $\tau \to -1$, σ approaches ∞ and the CES function turns to a linear function if $v=1$. i.e. $\sigma = 1/(1+(-1)) = \infty$ (undefined), where labour and capital are extreme substitutes:This third part is shown below:

Recall the CES model in equation (3.110):

$$y = \gamma\left[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}} e^u$$

Substituting for $v=1$, $\tau=-1$ and simplify;

$$y = \gamma\left[\delta x_1^{-(-1)} + (1-\delta)x_2^{-(-1)}\right]^{-\frac{1}{(-1)}} e^u$$

$$y = \gamma\left[\delta x_1 + (1-\delta)x_2\right]e^u$$

$$y = [\gamma\delta x_1 + \gamma(1-\delta)x_2]e^u$$

Let $\varphi_1 = \gamma\delta$ and $\varphi_2 = \gamma(1-\delta)$, then; $y = [\varphi_1 x_1 + \varphi_2 x_2]e^u$, taking the log of both sides

$$\ln y = \ln[\varphi_1 x_1 + \varphi_2 x_2] + u \qquad (3.112)$$

Since, in equation (3.112) the CES function is still nonlinear in parameters and not analytical despite taking the logarithm of both sides, it is not possible to estimate with the usual linear techniques.

Therefore, the CES function is often approximated by the "Kmenta approximation" Kmenta (1967) which is then estimated by linear estimation techniques.

Substituting where necessary in the Taylor series approximation in equation (3.35) from the classical approach section, recall the Kmenta approximation below;

$$\ln y = \varphi_0 + \varphi_1 \ln x_1 + \varphi_2 \ln x_2 + \frac{1}{2} B_{11} (\ln x_1)^2 + \frac{1}{2} B_{22} (\ln x_2)^2 + B_{12} \ln x_1 \ln x_2 \qquad (3.113)$$

Where; $\gamma = \exp(\varphi_0)$, $v = \varphi_1 + \varphi_2$, $\delta = \frac{\varphi_1}{\varphi_1 + \varphi_2}$, $\tau = \frac{B_{12}(\varphi_1 + \varphi_2)}{\varphi_1 \varphi_2}$, and $B_{12} = -B_{11} = -B_{22}$

For the purpose of this study, equation (3.113) can be simplified as

$$y^* = \varphi_0 + \varphi_1 x_1{}^* + \varphi_2 x_2{}^* + \varphi_{11} x_3{}^* + \varphi_{22} x_4{}^* + \varphi_{12} x_5{}^* \qquad (3.114)$$

Where; $y^* = \ln y$, $x_1{}^* = \ln x_1$, $x_2{}^* = \ln x_2$, $x_3{}^* = \frac{1}{2}(\ln x_1)^2$, $x_4{}^* = \frac{1}{2}(\ln x_2)^2$ and $x_5{}^* = \ln x_1 \ln x_2$ in order to have a model of the form below which can be solved using matrix techniques.

$$y^* = x_i{}^* \lambda + e_i{}^* \qquad (3.115)$$

Therefore, $E[y^*] = x_i{}^* \lambda = f(X, \gamma)$, the mean of eqn (3.115) where; $\lambda' = [\varphi_0\ \varphi_1\ \varphi_2\ B_{11}\ B_{22}\ B_{12}]$, $x_i{}^*$ is the design matrix of N x K elements and the error component is well behaved, i.e. $e_i{}^* \sim N(0, h^{-1} I_N)$, now the methodology of the OLS can be used directly to obtain the parameters of equation (3.113) then substituted to their respective representations ii equation (3.113) to get the values of the estimates of the model, but of concern in this study is the Bayesian approach.

Note: in equation (3.114) only the variables changed for convenience.

The following formulae is used at the likelihood level

$s^2 = \frac{(y^* - x^*\hat{\lambda})'(y^* - x^*\hat{\lambda})}{v}$, $v = N - K$, N is the sample size and K is the number of parameters.

$\hat{\lambda} = (x^{*\prime} x^*)^{-1} x^* y^*$, where; x^* the design matrix and y^* response variables

3.2.10 The Likelihood Function

Since the form of $f(X, \gamma)$ is known to be $f(x^*, \lambda)$ which is linear in parameters and variables, then from equation (3.115) the likelihood function for this study becomes

$$p(y^*/\lambda, h) = \frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} \left\{ \exp\left[-\frac{h}{2} \left\{ y - x^*\hat{\lambda} \right\}' \{ y - x^*\hat{\lambda} \} \right] \right\} \qquad (3.116)$$

Where; $\hat{\lambda} = (x^{*\prime} x^*)^{-1} x^* y^*$, x^* the design matrix and y^* response variables and $h = \frac{1}{\sigma^2}$ still the error precision.

3.2.11 The Prior

Like the Cobb-Douglas section, the Independent Normal-Gamma prior is also employed for this study depending on the form of $f(X, \gamma)$ which is only investigated using its non-informative aspect.

Therefore, from the law of independent random variables we have that

$P(\lambda, h) = P(\lambda) \cdot P(h)$

Where, $P(\lambda)$ is normal density and $P(h)$ is gamma density

$P(\lambda) = \frac{1}{(2\pi)^{\frac{k}{2}}} \left| \underline{V} \right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2} (\lambda - \underline{\lambda})' \underline{V}^{-1} (\lambda - \underline{\lambda}) \right]$

and $P(h)=C_G^{-1}h^{\frac{\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right)$

Where, C_G^{-1} is an integrating constant, It is deduced that: $E[\lambda/y^*]=\underline{\lambda}$ is the prior mean of λ and $Var(\lambda/h)=\underline{V}$ is the prior covariance matrix of λ, the mean of h, is $\underline{s}^{-2}$ and $\underline{v}$ is the degree of freedom.

3.2.12 The Posterior

Let the posterior (which is proportional to prior times likelihood) be denoted by $P(\lambda,h/y^*)$.

Mathematically, using $P(\lambda,h/y^*)=P(y^*/\lambda,h)\cdot P(\lambda)\cdot P(h)$,

But note that $P(\lambda,h/y^*)\neq P(\lambda/y^*,h)\cdot P(h/y^*,\lambda)$

Then, the posterior:

$$P(\lambda,h/y^*)=\frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\exp\left[-\frac{h}{2}\left(y^*-x^*\hat{\lambda}\right)'\left(y^*-x^*\hat{\lambda}\right)\right]\cdot\frac{1}{(2\pi)^{\frac{k}{2}}}|\underline{V}|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}(\lambda-\underline{\lambda})'\underline{V}^{-1}(\lambda-\underline{\lambda})\right]\cdot C_G^{-1}h^{\frac{\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right)$$

$$P(\lambda,h/y^*)\propto\exp\left[-\frac{1}{2}\{h\left(y^*-x^*\hat{\lambda}\right)'\left(y^*-x^*\hat{\lambda}\right)+(\lambda-\underline{\lambda})'\underline{V}^{-1}(\lambda-\underline{\lambda})\}\right]\cdot h^{\frac{N+\underline{v}-2}{2}}\exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}}\right)\quad(3.117)$$

This joint posterior density for λ and h does not take any well-known distributional form; so it cannot be solved analytically but only through a posterior simulation method.

The matrix multiplication to simplify the posterior distribution

The methodology involved in the Cobb-Douglas section follows according but only with change of parameters and variables for this section. The expression from equation (3.117) can be given as:

$$\begin{aligned}&h(y^*-x^*\lambda)'(y^*-x^*\lambda)+(\lambda-\underline{\lambda})'\underline{V}^{-1}(\lambda-\underline{\lambda})\\&=h\left(y^{*\prime}y^*-2\lambda'x^{*\prime}y^*+\lambda x^{*\prime}x^*\lambda\right)+(\lambda-\underline{\lambda})'\underline{V}^{-1}(\lambda-\underline{\lambda})\end{aligned}$$

$$= hy^{*\prime} y^* - 2h\lambda' x^{*\prime} y^* + h\lambda' x^{*\prime} x^* \lambda + \lambda' \underline{V}^{-1} \lambda - 2\lambda' \underline{V}^{-1} \underline{\lambda} + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda}$$

$$= hy^{*\prime} y^* + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda} + \lambda' (\underline{V}^{-1} + hx^{*\prime} x^*) \lambda - 2\lambda' (hx^{*\prime} y^* + \underline{V}^{-1} \underline{\lambda}) \qquad (3.118)$$

Let $\overline{V} = (\underline{V}^{-1} + hx^{*\prime} x^*)^{-1}$ (3.119) $\Rightarrow \overline{V}^{-1} = (\underline{V}^{-1} + hx^{*\prime} x^*)$ (3. 120)

$$\overline{\beta} = \overline{V}(hx^{*\prime} y^* + \underline{V}^{-1} \underline{\lambda}) \qquad (3.121)$$

Hence, substituting equation (3,119), (3.120) and (3.121) to have,

$$= hy^{*\prime} y^* + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda} + \lambda' \overline{V}^{-1} \lambda - 2\lambda' \overline{V}^{-1} \overline{\lambda} \qquad (3.122)$$

carrying out a simple mathematical assumption by including $-\overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$ and $+\overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$ into the equation (3.122) which does not change anything in the equation but help to achieve the desired result and have;

$$= hy^{*\prime} y^* + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda} + \lambda' \overline{V}^{-1} \lambda - 2\lambda' \overline{V}^{-1} \overline{\lambda} - \overline{\lambda}' \overline{V}^{-1} \overline{\lambda} + \overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$$

$$= hy^{*\prime} y^* + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda} - \overline{\lambda}' \overline{V}^{-1} \overline{\lambda} + \lambda' \overline{V}^{-1} \lambda - 2\lambda' \overline{V}^{-1} \overline{\lambda} + \overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$$

Let, $Q = hy^{*\prime} y^* + \underline{\lambda}' \underline{V}^{-1} \underline{\lambda} - \overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$ and

$$(\lambda - \overline{\lambda})' \overline{V}^{-1} (\lambda - \overline{\lambda}) = \lambda' \overline{V}^{-1} \lambda - 2\lambda' \overline{V}^{-1} \overline{\lambda} + \overline{\lambda}' \overline{V}^{-1} \overline{\lambda}$$

$$\Rightarrow h(y^* - x^* \lambda)'(y^* - x^* \lambda) + (\lambda - \underline{\lambda})' \underline{V}^{-1} (\lambda - \underline{\lambda}) = (\lambda - \overline{\lambda})' \overline{V}^{-1} (\lambda - \overline{\lambda}) + Q \qquad (3.123)$$

Substituting the expression in equation (3.123) for the expression in equation (3.117), to obtain

$$P(\lambda, h / y^*) \propto \exp\left[-\frac{1}{2} \left\{ \left(\lambda - \overline{\lambda}\right)' \overline{V}^{-1} \left(\lambda - \overline{\lambda}\right) \right\} \right] \cdot \exp[-\frac{1}{2} Q] h^{\frac{N+\underline{v}-2}{2}} \exp\left(\frac{-h\underline{v}}{2\underline{s}^{-2}} \right) \qquad (3.124)$$

By ignoring the terms that do not involve λ in eqn (3.124) to obtain,

$$P(\lambda / y^*, h) \propto \exp\left[-\frac{1}{2} \left\{ \left(\lambda - \overline{\lambda}\right)' \overline{V}^{-1} \left(\lambda - \overline{\lambda}\right) \right\} \right] \qquad (3.125)$$

Which implies that $\lambda / y, h \sim N(\overline{\lambda}, \overline{V})$, a **Multivariate Normal density**

Where, $\overline{V} = (\underline{V}^{-1} + hx^{*\prime} x^*)^{-1}$ and $\overline{\lambda} = \overline{V}(hx^{*\prime} y^* + \underline{V}^{-1}\underline{\lambda})$

Similarly, by treating equation (3.124) as a function of h ignoring terms that do not involve h one can obtain

$$P\left(h / y^*, \lambda\right) \propto h^{\frac{N+\underline{v}-2}{2}} \exp[-\frac{h}{2}\{(y^* - x^*\lambda)'(y^* - x^*\lambda) + \underline{v}\underline{s}^2\}] \qquad (3.126)$$

This also implies that $h / y^*, \lambda \sim G(\overline{s}^{-2}, \overline{v})$, a **Gamma density**

Where, $\overline{v} = N + \underline{v}$ and $\overline{s}^{2} = \dfrac{(y^* - x^*\lambda)'(y^* - x^*\lambda) + \underline{v}\underline{s}^2}{\overline{v}}$

The formulae of equation (3.125) and (3.126) are familiar to those of the conjugate normal-gamma priors now but do not relate directly to the posterior of interest. Therefore, the conditional posteriors in equations (3.125) and (3.126) do not directly tell us everything about the posterior, $P\left(\lambda, h / y^*\right)$.

After obtaining the values of the posterior estimates, then substitute where necessary to obtain the real estimates of the model.

3.2.13 Metropolis-Within-Gibbs

Metropolis-Hastings Algorithm incorporated within a Gibbs sampler to draw samples from the parameters whose full conditional probability density function cannot be analytically determined.The algorithm gives a powerful posterior simulator for $P\left(\beta \mid y\right)$ with which posterior conditionals used in the Gibbs Sampler that involvles more than two blocks was introduced. Therefore, it should be noted that in some models it is not an easy task to draw directly from $P(\beta \mid y)$ as it may seem in this study.

However, it can be shown that, if we use a Metropolis-within-Gibbs algorithm for $P\left(\beta \mid y, h\right)$, the resulting draws, $\beta^{(s)}$ and $h^{(s)}$ for $s = 1, 2, \cdots, S$ are valid posterior simulator draws.

Since, the posterior conditionals arrived at in this study has no convenient form and the marginal likelihood are undefined, we then introduce the Metropolis-Hastings algorithm; particularly, the Metropolis within-Gibbs.

3.2.14 Steps involved in the Metropolis-within-Gibbs algorithm

Step 1: choose initial values for $\beta^{(0)}$,

Step2: Generate x^* from the conditional of $\beta / y, h$

Step 3: set the acceptance ratio $\alpha = \min\{1, \frac{f(x^*)}{f(\beta^{(0)})}\}$

Step 4: Sample U from a standard uniform distribution i.e. $U[0,1]$

Step 5: if $U \leq \alpha$ set $x_1 = x^*$. If not set $x_1 = \beta^{(0)}$

Step 6: repeat the above steps until the desired result is achieved.

3.10 The Metropolis–Hastings algorithm

The Metropolis–Hastings algorithm works by generating a sequence of sample values in such a way that, as more and more sample values are produced, the distribution of values more closely approximates the desired distribution, *P(x)*. These sample values are produced iteratively, with the distribution of the next sample being dependent only on the current sample value (thus making the sequence of samples into a Markov Chain). Specifically, at each iteration, the algorithm picks a candidate for the next sample value based on the current sample value. Then, with some probability, the candidate value is either accepted (in which case the candidate value is used in the next iteration) or rejected (in which case the candidate value is discarded, and current value is reused in the next iteration) the probability of acceptance is determined by comparing the values of the function *f(x)* of the current and candidate sample values with respect to the desired distribution *P(x)*.

3.2.18 Burn-In

This is a process of discarding the first S_0 of the S replications, by which S_0 is removed because the time it takes for the chain to converge varies depending on the starting point, while the remaining draws say; S_1 is retained for the estimate of $E[g(\beta,h)/y]$, (posterior mean of β *and* h) where $S_0+S_1=S$. This is usually done to make sure that the draws which do not converge are being eliminated from the Metropolis-within Gibbs algorithm in order for our draws to get closer to the stationary distribution of interest and less dependent on the starting point.

Note:

$\hat{g}s_1=\frac{1}{s_1}\sum_{s=S_0+1}^{S} g((\beta,h)^{(S)})$ Converges to $E[g(\beta,h)\mid y]$ as S_1 tends to infinity.

Where, S is the number of draws taken, S_0 are the draws discarded due to the effects of the initial values, S_1 are the retained draws after burn in, $\hat{g}s_1$ is the corresponding estimates of the posterior mean, $E[g(\beta,h)\mid y]$ after burn-in and $g(\beta,h)^{(S)}$ is the function of interest at S draws.

3.12 Geweke Convergence Diagnostic

The Geweke convergence diagnostic test is one of the diagnostic tools to check if the Metropolis-Hasting algorithm has converged. The Geweke diagnostic takes two overlapping parts (usually the first 0.1 and last proportions) of the Markov chain and compares the means of both parts, using a difference of means test to see if the two parts of the chain are from the same distribution (null hypothesis).

The test statistics is a standard Z-score with the standard errors adjusted for autocorrelation. It is known that if Convergence Diagnostics (CD) is less than 1.96 in absolute value, then conclude that the convergence of the Markov Chain Monte Carlo (MCMC) algorithm has occurred.

3.2.19 Numerical Standard Error

Numerical Standard Error is defined as a measure of approximation error which captures simulation error of the mean rather than posterior uncertainty.

$$\sqrt{S}\left\{\hat{g}s - E\left[g(\beta,h)\mid y\right]\right\} \rightarrow N(0,\sigma_g^2) \qquad (3.18)$$

As S goes to infinity, where $\sigma_g^2 = \mathrm{var}\left[g(\beta,h)\mid y\right]$, the variance of β *and* h,

Using the fact that the Standard normal has 95% of its probability located within 1.96 standard deviations from its mean yields the approximate result that:

$$Pr\left[-1.96\frac{\sigma_g}{\sqrt{S}} \leq \hat{g}s - E\left[g(\beta,h)\mid y\right] \leq 1.96\frac{\sigma_g}{\sqrt{S}}\right] = 0.95$$

By controlling S, the $\hat{g}s - E[g(\beta,h)\mid y]$ is sufficiently small with a high degree of probability, The term $\frac{\sigma_g}{\sqrt{S}}$ is known as the numerical standard error.

3.2.20 Data Generation Process (DGP)

In attempt to estimate the parameters of Bayesian nonlinear production function with error specifications. We adopt Markov-Chain Monte Carlo (MCMC) experiments. The Monte Carlo simulation technique was generated using the following:

i. Consider the nonlinear model of the form

$$Y_i = f(X_i,\beta) + \varepsilon_i \quad i = 1\ldots n \qquad \varepsilon \sim N(0,\sigma^2)$$

Where Y_i are responses, f is a known function of covariate vector explanatory variable(s), $X_j = (x_1,\ldots,x_k)^T$ and the parameter vector $\beta = (\beta_i,\ldots,\beta_p)^T$ is the coefficient and ε_i are assumed to be uncorrelated with mean zero and constant variance.

ii. fixed the arbitrary values for the regression coefficient, for all the models i.e for Cobb Douglas with $\beta_i, i = 0,1, and\ 2$ and for CES with $\beta_i, i = 0,1,2,3 and\ 4$ and for the intercept $\beta_0\ (X_0 = 1)$

iii. generated the explanatory variables X_{ij} from uniform $U[0.1]$ distribution i.e $X_{ij} \sim U[0,1],\ i = 0,1,2\ \ j = 1,2,...,N.$

iv. Also, generated the error terms ε_i from a standard normal distribution with $E(\varepsilon_i) = 0$ and $E(\varepsilon_i^2) = 1$ for $i = 1,2,\ldots,n$ $i.e\ \varepsilon_i \sim N(0,1)$ in each of the model

v. Incorporated all the above steps in the models considered to obtain the dependent variable (data of interest), $Y_j,\ j = 1,2,3,...,N$ with varying sample sizes. N = 50, 100, 150, 250 and 500.

vi. Finally, the Bayesian approached was introduced to obtain the estimates of the models at different sample sizes stated above. However, since the joint posterior densities for the three models did not take any well-known distributional form, their respective Posterior estimates were not easy to compute due to its numerical intractability and this consequentially lead us to conditional marginal posterior distribution whereby $p\left(\hat{\beta}/y^{*},h\right) \sim N\left(\overline{\beta},\overline{V}\right)$ and $p\left(h/y^{*},\hat{\beta}\right) \sim G\left(\overline{s}^{-2},\overline{v}\right)$ for Cobb Douglas production function with Multiplicative error model, $p\left(\hat{\beta}/y^{**},h\right) \sim N\left(\overline{\beta},\overline{V}\right)$ and $p\left(h/y^{**},\hat{\beta}\right) \sim G\left(\overline{s}^{-2},\overline{v}\right)$ for Cobb Douglas production function with additive error model and $p\left(\lambda/y^{*},h\right) \sim N\left(\overline{\lambda},\overline{V}\right)$ and $p\left(h/y^{*},\lambda\right) \sim G\left(\overline{s}^{-2},\overline{v}\right)$ for CES production function which were achieved by using independent Normal Gamma prior via Metropolis-Within-Gibbs sampler.

The prior specification for β, h and v of three models for variance covariance matric for β_i, the prior variance of the model's residual and the prior degree of freedom respectively shall be elicited by noninformative prior. A noninformative prior is one assign equal probabilities to all possible states of the parameter space with the aim of rectifying the subjectivity problem which nullifies the benefit of Bayesian analysis because it reduces the latter to an inference based only on the likelihood. In one sense, this noninformative prior has very attractive properties and, given the close relationship with OLS results, provides a bridge between the Bayesian and frequentist approaches (Koop, 2003). This is why there is a strong objection to the practice of noninformative priors. In this study we set v small relatively to N and V to a large value which confirm that the prior information plays little role in the posterior and the prior distribution is independent Normal Gamma prior.

CHAPTER FOUR

4.0 Analysis of Data and Interpretation

In this chapter, the estimated paremeter results for Cobb-Douglas production functions with additive and multiplicative models, and CES production for classical using Gauss Newton Method(GNM) and Bayesian approached were reported

4.1 Cobb Douglas Production Functions with Additive and Multiplicative Error

The regression coefficients β_i, $i = 0,1,2$ where β_0 is intercept, β_1, β_2 are the coefficient of capital and labour respectively for three scenarios considered under C-D production functions when $\beta_1 + \beta_2 = 1$ $(\beta_0 = 15, \beta_1 = 0.85, \beta_2 = 0.15)$ is known as Constant Return to Scale, $\beta_1 + \beta_2 < 1$ $(\beta_0 = 15, \beta_1 = 0.65, \beta_2 = 0.15)$ is the Decreasing Return Scale and $\beta_1 + \beta_2 > 1$ $(\beta_0 = 15, \beta_1 = 0.90, \beta_2 = 0.20)$ is the Increasing Return Scale.The explanatory variables were drawn independently from a uniform [0,1] distribution, $X_{ij} \sim U[0,1]$, $i = 0,1,2$ and $j = 1,2,...,N.$ when the errors were drawn from a standard normal distribution, $\varepsilon_j \sim N(0,1)$, incorporated into non- linear cobb-Douglas model to obtain the response variable (data of interest) i.e $y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} e^u$ and $y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} + u_i$. The Bayesian log normal linear regression model was fitted using independent Normal Gamma prior and achieved by using the non-informative prior which is the inverse of precision (h) prior i.e. $p(\beta, h) \alpha \frac{1}{h}$. The prior used are 0.001 (500.40), 0.005 (90.7), 0.01 (449.097) under the various sample sizes of N=50, N=100, N=150, N=250 and N= 500 with the standard deviations (SD) in parenthesis. Direct sampling from the joint posterior

$p(\beta / y, h)$ is achieved using the posterior simulator technique called metropolis- within-Gibbs algorithm, which was used to obtain the posterior estimates by setting up a 10,000 iterations and a burn-in of 1000 to attain convergence of the posterior estimates under MCMC technique.

4.2 CES Production Function

The data used for the analysis were generated using MCMC technique. The explanatory variables X_1, X_2 are independently drawn from a uniform distribution [0, 1] i.e. $X_{ij} \sim U[0,1]$. The values for the regression coefficients were fixed which for the first scenario are $\gamma(1.0)$, $\delta(0.6)$, $\tau(0.5)$ and $\nu(1.1)$, for second scenario $\gamma(1.0)$, $\delta(0.6)$, $\tau(-0.5)$ and $\nu(1.1)$, for third scenario $\gamma(1.0)$, $\delta(0.6)$, $\tau(0)$ and $\nu(1.1)$. Where γ is the factor of production, δ is the distribution parameters between zero and one and can be used to determine factor shares, τ is the substitution parameter in which value of elasticity of substitution (σ) was derived. The error incorporated in CES MODEL was drawn from a standard normal distribution, $\varepsilon_j \sim N(0,1)$ to obtain the response variable (data of interest) i.e $y = \gamma\left[\delta x_1^{-\rho} + (1-\delta)x_2^{-\rho}\right]^{-\frac{v}{\rho}} e^u$. The prior used are 0.001 (500.40), 0.005 (90.7), 0.01 (449.097) under the various sample sizes of N=50, N=100, N=150, N=250 and N= 500 with the standard deviations (SD) in parenthesis, a metropolis Hasting Within Gibbs algorithm was used with the Normal-Gamma prior to obtain the posterior estimates. The MCMC of 10,000 with burn-in of 1000 were used to attain convergence of the posterior estimates.

Table 1: Estimates of the Additive and Multiplicative Error - Based Cobb-Douglas Production Function; For Constant Return to Scale ($\beta_1 + \beta_2 = 1$), N=50

		Additive Error		**Multiplicative Error**	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
50	β_0	15.445690 **(**0.446410**)**	15.815700 **(**0.000621**)**	10.879500 (0.221240**)**	10.882991 **(**0.000595**)**
	β_1	0.845610 **(**0.046720**)**	0.934454 **(**0.000372**)**	0.829240 (0.119940)	0.829921 **(**0.000317**)**
	β_2	0.186740 (0.026600)	0.160182 (0.000476)	0.1811300 (0.091720)	0.180948 (0.000247)

Table 1 shows estimates of additive and multiplicative Error- Based model in classical and Bayesian approaches under constant return to scale with sample size of 50. For GNM, the estimates of β_1 and β_2 are (0.845610, 0.829240) and (0.186740, 0.1811300) with standard errors (0.046720, 0.119940) and (0.026600, 0.091720) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.934454, 0.829921) and (0.160182, 0.180948) with standard errors (0.000372, 0.000317) and (0.000476, 0.000247) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error- based model. Hence, Bayesina estimation of Multiplicative error based C-D for Constant returns to scale should be preferred.

Table 2: Estimates of the Additive and Multiplicative Error - Based Cobb-Douglas Production Function; For Constant Return to Scale ($\beta_1 + \beta_2 = 1$), N=100

		Additive Error		**Multiplicative Error**	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
100	β_0	14.749440 (0.289740)	15.69809 (0.000445)	12.723619 (0.170030)	12.726243 (0.000426)
	β_1	0.832800 (0.030560)	0.994454 (0.000291)	0.791700 (0.108620)	0.792139 (0.000267)
	β_2	0.128510 (0.015610)	0.233634 (0.000299)	0.174650 (0.084730)	0.174478 (0.000213)

Table 2 shows estimates of additive and multiplicative Error- Based model in classical and Bayesian approaches under constant return to scale with sample size of 100. For GNM, the estimates of β_1 and β_2 are (0.832800 , 0.791700) and (0.128510, 0.174650) with standard errors (0.030560, 0.108620) and (0.015610, 0.084730) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.994454, 0.792139) and (0.233634, 0.174478) with standard errors (0.000291, 0.000267) and (0.000299, 0.000213) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error- based model. Hence, Bayesina estimation of Multiplicative error based C-D for Constant returns to scale should be preferred.

Table 3: Estimates of the Additive and Multiplicative Error - Based Cobb-Douglas Production Function; For Constant Return to Scale ($\beta_1 + \beta_2 = 1$), N=150

		Additive Error		**Multiplicative Error**	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
150	β_0	14.990990 (0.287480)	15.086090 (0.000382)	14.201640 (0.135700)	14.203580 (0.000337)
	β_1	0.853670 (0.027170)	0.856928 (0.000233)	0.823940 (0.091980)	0.824290 (0.000224)
	β_2	0.151210 (0.013640)	0.180898 (0.000248)	0.196540 (0.076410)	0.196383 (0.000190)

Table 3 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under constant return to scale with sample size of 150. For GNM, the estimates of β_1 and β_2 are (0.853670, 0.823940) and (0.151210, 0.196540) with standard errors (0.027170,0.091980) and (0.013640, 0.076410) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.856928, 0.824290) and (0.180898, 0.196383) with standard errors (0.000233, 0.000224) and (0.000248, 0.000190) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesina estimation of Multiplicative error based C-D for Constant returns to scale should be preferred.

Table 4: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas ProductionFunction; For Constant Return to Scale ($\beta_1 + \beta_2 = 1$), N=250

		Additive Error		Multiplicative Model	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
250	β_0	15.038910 (0.221834)	15.67601 (0.000291)	13.243674 (0.107760)	13.24594 (0.000269)
	β_1	0.841602 (0.020145)	0.837894 (0.000195)	0.806600 (0.068750)	0.806481 (0.000172)
	β_2	0.159087 (0.009644)	0.159936 (0.000155)	0.157030 (0.062110)	0.157264 (0.000152)

Table 4 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under constant return to scale with sample size of 250. For GNM, the estimates of β_1 and β_2 are (0.841602, 0.806600) and (0.159087, 0.157030) with standard errors (0.020145, 0.068750) and (0.009644, 0.062110) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.837894, 0.806481) and (0.159936, 0.157264) with standard errors (0.000195, 0.000172) and (0.000155, 0.000152) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesina estimation of Multiplicative error based C-D for Constant returns to scale should be preferred.

Table 5: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function; For Constant Return to Scale ($\beta_1 + \beta_2 = 1$), N=500

		Additive Error		**Multiplicative Error**	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
500	β_0	14.976400 (0.137190)	13.573850 (0.000197)	15.170155 (0.084730)	15.172110 (0.000186)
	β_1	**0.841800 (0.014130)**	**0.759519 (0.000120)**	**0.845710 (0.048920)**	**0.845861 (0.000114)**
	β_2	**0.148060 (0.007060)**	**0.132379 (0.000116)**	**0.176550 (0.049910)**	**0.176499 (0.000103)**

Table 5 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under constant return to scale with sample size of 500. For GNM, the estimates of β_1 and β_2 are (0.841800, 0.845710) and (0.148060, 0.176550) with standard errors (0.014130, 0.04892) and (0.007060, 0.049910) respectively. For Bayesian,the estimates of β_1 and β_2 are (0.759519, 0.845861) and (0.132379, 0.176499) with standard errors (0.000120, 0.000114) and (0.000116, 0.000103) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesina estimation of Multiplicative error based C-D for Constant returns to scale should be preferred.

Table 6: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function; For Increasing Return to Scale ($\beta_1+\beta_2>1$), N=50

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	15.560690 (0.505580)	14.224050 (0.000632)	10.845780 (0.236300)	10.851550 (0.000621)
	β_1	0.943790 (0.054560	0.790109 (0.000369)	0.877700 (0.134900)	0.877539 (0.000364)
50	β_2	0.194150 (0.029130)	0.217897 (0.000479)	0.229200 (0.092000)	0.229505 (0.000247)

Table 6 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under increasing return to scale with sample size of 50. For GNM, the estimates of β_1 and β_2 are (0.943790, 0.877700) and (0.194150, 0.229200) with standard errors (0.054560, 0.134900) and (0.029130, 0.092000) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.790109, 0.877539) and (0.217897, 0.229505) with standard errors (0.000369, 0.000364) and (0.000479, 0.000247)respectively. The findings implies that additive error based performs than multiplicative error based model in classical contrary to Bayesian approach where multiplicative error based performs better than additive error based model with reduction in their standard errors. Hence, Bayesianestimation of Multiplicative error based C-D for Increasing returns to scale should be preferred.

Table 7: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas ProductionFunction; For Increasing Return to Scale ($\beta_1+\beta_2>1$), N=100

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	15.086820 (0.332270)	14.207920 (0.000466)	13.86358 (0.187020)	13.867410 (0.000442)
	β_1	0.908400 (0.036290)	0.868892 (0.000294)	0.928510 (0.102950)	0.928892 (0.000256)
100	β_2	0.210600 (0.018960)	0.193729 (0.000299)	0.221600 (0.084660)	0.221449 (0.000221)

Table 7 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under increasing return to scale with sample size of 100. For GNM, the estimates of β_1 and β_2 are (0.908400, 0.928510) and (0.210600, 0.221600) with standard error (0.036290, 0.102950) and (0.018960, 0.084660) respectively. For Bayesian, (0.868892, 0.928892) and (0.193729, 0.221449) with standard error (0.000294, 0.000256) and (0.000299, 0.000221)respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesian estimation of Multiplicative error based C-D for Increasing returns to scale should be preferred.

Table 8: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function; For Increasing Return to Scale ($\beta_1 + \beta_2 > 1$), N=150

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	15.496290 (0.328870)	13.140370 (0.000379)	15.51768 (0.150380)	15. 520540 (0.000372)
	β_1	0.904340 (0.031380)	0.741109 (0.000232)	0.961380 (0.087520)	0.9616863 (0.000216)
150	β_2	0.230670 (0.016180)	0.188109 (0.000246)	0.248360 (0.076000)	0.2482154 (0.000189)

Table 8 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under increasing return to scale with sample size of 150. For GNM, the estimates of β_1 and β_2 are (0.904340, 0.961380) and (0.230670, 0.248360) with standard errors (0.031380, 0.087520) and (0.016180, 0.076000) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.741109, 0.9616863) and (0.188109, 0.2482154) with standard errors (0.000232, 0.000216) and (0.000246, 0.000189)respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model.Hence, Bayesian estimation of Multiplicative error based C-D for Increasing returns to scale should be preferred.

Table 9: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas ProductionFunction; For Increasing Return to Scale ($\beta_1 + \beta_2 > 1$), N=250

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	14.873640 (0.265960)	13.970310 (0.000290)	15.933490 (0.112360)	13.935450 (0.000280)
	β_1	0.862800 (0.024340)	0.839516 (0.000194)	0.910720 (0.070780)	0.910978 (0.000176)
250	β_2	0.207460 (0.012090)	0.193333 (0.000158)	0.206170 (0.062180)	0.206048 (0.000155)

Table 9 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under increasing return to scale with sample size of 250. For GNM, the estimates of β_1 and β_2 are (0.862800, 0.910720) and (0.207460, 0.206170)with standard errors (0.024340,0.070780) and (0.012090, 0.062180) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.839516, 0.910978) and (0.193333, 0.206048) with standard errors (0.000194, 0.000176) and (0.000158, 0.000155) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicativeerror based model.Hence, Bayesian estimation of Multiplicative error based C-D for Increasing returns to scale should be preferred.

Table 10: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function; For Increasing Return to Scale ($\beta_1 + \beta_2 > 1$), N=500

		Additive		**Multiplicative**	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
500	β_0	15.096986 (0.158986)	14.537930 (0.000184)	15.152410 (0.083210)	15.154280 (0.000146)
	β_1	**0.916189** **(0.017134)**	**0.897875** **(0.000118)**	**0.894210** **(0.049910)**	**0.894150** **(0.000116)**
	β_2	**0.201102** **(0.008889)**	**0.204861** **(0.000114)**	**0.226870** **(0.049800)**	**0.227022** **(0.000102)**

Table shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under increasing return to scale with sample size of 500. For GNM, the estimates of β_1 and β_2 are (0.916189, 0.894210) and (0.201102, 0.226870) with standard error (0.017134, 0.049910) and (0.008889,0.049800) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.897875, 0.894150) with standard error(0.000118, 0.000116) and (0.000114, 0.000102) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicativeerror based model. Hence, Bayesian estimation of Multiplicative error based C-D for Increasing returns to scale should be preferred.

Table 11: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function For Decreasing Return to Scale ($\beta_1 + \beta_2 < 1$), N=50

		Additive		**Multiplicative**	
Sample Size	**Estimate**	**GNMAE**	**BEAE**	**GNMME**	**BEME**
	β_0	14.665940 (0.416250)	14.678560 (0.000651)	10.109150 (0.240500)	10.113200 (0.000644)
	β_1	0.631070 (0.030760)	0.632867 (0. 000408)	0.622100 (0.134900)	0.626600 (0.000361)
50	β_2	0.126830 (0.018400)	0.136904 (0.000327)	0.129000 (0.120300)	0.1288124 (0.000322)

Table 11 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under decreasingreturn to scale with sample size of 50. For GNM, the estimates of β_1 and β_2 are (0.631070, 0.622100) and (0.126830, 0.129000) with standard errors (0.030760, 0.134900) and (0.018400, 0.120300) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.632867, 0.626600) and (0.136904, 0.1288124) with standard errors (0. 000408, 0.000361) and (0.000327, 0.000322) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesian estimation of Multiplicative error based C-D for Decreasing returns to scale should be preferred.

Table 12: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function For Decreasing Return to Scale ($\beta_1 + \beta_2 < 1$), N=100

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNMAE**	**BEAE**	**GNMME**	**BEME**
	β_0	15.150520 (0.381290)	14.118910 (0.000463)	12.727310 (0.185690)	12.731500 (0.000452)
	β_1	0.669610 (0.028520)	0.633128 (0.000256)	0.673300 (0.102440)	0.673679 (0.000254)
100	β_2	0. 144800 (0.020580)	0. 154247 (0.000318)	0.096720 (0.107990)	0.0965238 (0.000270)

Table 12 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under decreasing return to scale with sample size of 100. For GNM, the estimates of β_1 and β_2 are (0.669610, 0.673300) and (0. 144800,0.096720) with standard errors (0.028520, 0.102440) and (0.020580,0.107990)respectively. For Bayesian, the estimates of β_1 and β_2 are (0.633128, 0.673679) and (0. 154247,0.0965238) with standard errors (0.000256, 0.000254) and (0.000318, 0.000270) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesian estimation of Multiplicative error based C-D for Decreasing returns to scale should be preferred.

Table 13: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function For Decreasing Return to Scale ($\beta_1 + \beta_2 < 1$), N=150

		Additive		**Multiplicative**	
Sample Size	**Estimate**	**GNMAE**	**BEAE**	**GNMME**	**BEME**
	β_0	15.326870 (0.272750)	15.40288 (0.000358)	14.385300 (0.144610)	14.388400 (0.000353)
	β_1	0.662830 (0.022400)	0.702570 (0.000221)	0.706690 (0.087320)	0.707000 (0.000216)
150	β_2	0.142250 (0.013380)	0.165952 (0.000227)	0.130850 (0.091270)	0.130700 (0.000205)

Table 13 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under decreasing return to scale with sample size of 150. For GNM, the estimates of β_1 and β_2 are (0.662830, 0.706690) and (0.142250, 0.130850) with standard errors (0.022400, 0.087320) and (0.013380, 0.091270) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.702570, 0.707000) and (0.165952, 0.130700) with standard errors (0.000221, 0.000216) and (0.000227, 0.000205) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesian estimation of Multiplicative error based C-D for Decreasing returns to scale should be preferred.

Table 14: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production FunctionFor Decreasing Return to Scale ($\beta_1 + \beta_2 < 1$), N=250

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	14.837120 (0.198030)	15.075890 (0.000279)	13.284260 (0.111980)	13.2864 (0.000266)
	β_1	0.637683 (0.040720)	0. 642410 (0.000172)	0.660330 (0.070660)	0.660600 (0.000163)
250	β_2	0.131770 (0.02569)	0.162535 (0.000132)	0.106870 (0.068710)	0.106700 (0.000124)

Table 14 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under decreasing return to scale with sample size of 250. For GNM, the estimates of β_1 and β_2 are (0.637683, 0.660330) and (0.131770, 0.106870) with standard errors (0.040720, 0.070660) and (0.02569, 0.068710) respectively. For Bayesian, the estimates of β_1 and β_2 are (0. 642410, 0.6606) and (0.162535, 0.106700) with standard errors (0.000172, 0.000163) and (0.000132, 0.000124) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Among the C-D functions for decreasing returns to scale, the Bayesian estimation with multiplicative error outperforms with minimal standard errors

Table 15: Estimates of the Additive and Multiplicative Error Based Cobb-Douglas Production Function For Decreasing Return to Scale ($\beta_1 + \beta_2 < 1$), N=500

		Additive		Multiplicative	
Sample Size	**Estimate**	**GNM**	**Bayesian**	**GNM**	**Bayesian**
	β_0	16.062334 (0.190495)	15.383480 (0.000196)	14.656010 (0.084280)	14.657900 (0.000187)
	β_1	**0.521822 (0.014130)**	**0.682444 (0.000121)**	**0.643900 (0.050010)**	**0.644100 (0.000116)**
500	β_2	**0.126701 (0.008448)**	**0.1704460 (0.000114)**	**0.143590 (0.048900)**	**0.143500 (0.000101)**

Table 15 shows estimates of additive and multiplicative error- based model in Classical and Bayesian approach under decreasing return to scale with sample size of 500. For GNM, the estimates of β_1 and β_2 are (0.521822, 0.643900) and (0.126701, 0.143590) with standard errors (0.014130, 0.050010) and (0.008448, 0.048900) respectively. For Bayesian, the estimates of β_1 and β_2 are (0.682444, 0.6441) and (0.1704460, 0.143500) with standard errors (0.000121, 0.000116) and (0.000114, 0.000101) respectively. Bayesian method produces better estimates in C-D function for both additive and multiplicative error based model. Hence, Bayesian estimation of Multiplicative error based C-D for Decreasing returns to scale should be preferred.

Table 16: Summary for the two specification of errors of the C-D production function Constant Returns to Scale, with sample sizes (N); 50, 100, 150, 250 and 500

	Additive				Multiplicative			
	GNM		Bayesian		GNM		Bayesian	
Sample Size	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
50	0.046720	0.026600	0.000372	0.000476	0.119940	0.091720	0.000317	0.000247
100	0.030560	0.015610	0.000291	0.000299	0.108620	0.084730	0.000267	0.000213
150	0.027170	0.013640	0.000233	0.000248	0.091980	0.076410	0.000224	0.000190
250	0.020145	0.009644	0.000195	0.000155	0.068750	0.062110	0.000172	0.000152
500	0.014130	0.007060	0.000120	0.000116	0.04892	0.049910	0.000114	0.000103

Table 16. shows the standard errors of all the additive and multiplicative errors –based model in classical and Bayesian for constant returns to scale of cobb- Douglas production functions. The result shows that standard errors of the estimators decreased consistently as sample size increased i.e the GNM with additive errors when N=50 to 500, the standard errors for β_1 and β_2 are (0.046720, 0.030560, 0.027170, 0.020145, 0.014130) and (0.026600, 0.015610, 0.013640, 0.009644, 0.007060) respectively, while that of multiplicative errors for β_1 and β_2 are (0.119940, 0.108620, 0.091980, 0.068750, 0.04892) and (0.091720, 0.084730, 0.076410, 0.062110, 0.049910) respectively. The estimators of Bayesian with additive errors β_1 and β_2 are (0.000372, 0.000294, 0.000233, 0.000195, 0.000120) and (0.000476, 0.000301, 0.000248, 0.000155, 0.000116) respectively consistently decreases as sample size increases. The Bayesian approach with multiplicative errors is preferable to that of others for constant returns to scale.

Table 17: Summary for the two specification of errors of the C-D production functionIncreasing Returns to Scale, with sample sizes (N); 50, 100, 150, 250 and 500

	Additive				**Multiplicative**			
	GNM		**Bayesian**		**GNM**		**Bayesian**	
Sample Size	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
50	0.054560	0.029130	0.000369	0.000479	0.134900	0.092000	0.000364	0.000248
100	0.036290	0.018960	0.000294	0.000299	0.102950	0.084660	0.000256	0.000221
150	0.031380	0.016180	0.000232	0.000246	0.087520	0.076000	0.000216	0.000189
250	0.024340	0.012090	0.000194	0.000158	0.070780	0.062180	0.000176	0.000156
500	0.017134	0.008889	0.000118	0.000114	0.049910	0.049910	0.000116	0.000102

Table 17. shows the standard errors of all the additive and multiplicative errors –based model in classical and Bayesian for increasing returns to scale of cobb- Douglas production functions. The result shows that standard errors of the estimators decreased consistently as sample size increased i.e the GNM with additive errors when N=50 to 500, the standard errors for β_1 and β_2 are (0.054560,0.036290, 0.031380, 0.024340, 0.017134) and (0.029130, 0.018960, 0.016180, 0.012090, 0.008889) respectively, while that of multiplicative errors for β_1 and β_2 are (0.134900, 0.102950, 0.087520, 0.070780, 0.049910) and (0.092000, 0.084660, 0.076000, 0.062180, 0.049800) respectively. The estimators of Bayesian with multiplicative errors β_1 and β_2 are (0.000364,0.000256, 0.000216, 0.000176, 0.000116) and (0.000248,0.000221, 0.000189, 0.000156, 0.000102) respectively consistently decreases as sample size increases. The Bayesian approach with multiplicative errors is preferable to that of others for increasing returns to scale.

Table 18: Summary for the two specification of errors of the C-D production function Decreasing Returns to Scale, with sample sizes (N); 50, 100, 150, 250 and 500

	Additive				**Multiplicative**			
	GNM		**Bayesian**		**GNM**		**Bayesian**	
Sample Size	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
50	0.030760	0.018400	0. 000408	0.000327	0.134900	0.120300	0.000361	0.000322
100	0.028520	0.020580	0.000256	0.000318	0.102440	0.107990	0.000254	0.000270
150	0.022400	0.013380	0.000221	0.000227	0.087320	0.091270	0.000216	0.000205
250	0.04072	0.02569	0.000176	0.000172	0.070660	0.068710	0.000163	0.000132
500	0.014130	0.008448	0.000121	0.000114	0.050010	0.048900	0.000116	0.000101

Table 18. shows the standard errors of all the additive and multiplicative errors –based model in classical and Bayesian for decreasing returns to scale of cobb- Douglas production functions. The result shows that standard errors of the estimators decreased consistently as sample size increased i.e the GNM with additive errors when N=50 to 500, the standard errors for β_1 and β_2 are (0.030760,0.028520, 0.022400, 0.040720, 0.014130) and (0.018400, 0.020580, 0.013380, 0.02569, 0.008448) respectively, while that of multiplicative errors for β_1 and β_2 are (0.134900, , 0.102440, 0.087320, 0.070660,0.050010) and (0.120300, 0.107990, 0.091270, 0.068710,0.048900) respectively. The estimators of Bayesian with multiplicative errors β_1 and β_2 are (0.000361, 0.000254, 0.000216, 0.000163, 0.000116) and (0.000322, 0.000270, 0.000205, 0.000132, 0.000101) respectively consistently decreases as sample size increases. The Bayesian approach with multiplicative errors is preferable to that of others for decreasing returns to scale.

Table 19: The Multiplicative Error Based CES Production Function for Effect of Change in Inputs $\tau(0.5)$

Sample size	True value	GNM	Bayesian
50	γ (1.0)	3.792400 (2.286300)	1.586733 (0.008985)
100		10.1134 (2.180800)	0.892900 (0.004301)
150		2.172200 (0.463700)	0.813007 (0.003695)
250		1.801900 (0.312100)	0.972880 (0.002506)
500		1.319700 (0.117500)	0.850874 (0.001659)
50	δ (0.6)	0.822600 (0.344900)	0.491960 (0.012560)
100		0.056500 (0.069700)	0.239601 (0.007983)
150		0.531700 (0.129000)	0.522710 (0.006410)
250		0.864800 (7.197800)	0.450175 (0.002597)
500		0.348200 (0.277300)	0.468579 (0.001789)
50	τ (0.5)	2.601900 (6.297900)	0.954027 (0.005785)
100		5.358900 (4.963900)	1.069045 (0.003074)
150		-0.032100 (1.122100)	0.913035 (0.001808)
250		**67.437800 (2221.762600)**	**0.119093 (0.001380)**
500		2.926300 (5.476700)	0.468569 (0.000817)
50	ν (1.1)	2.748300 (1.699000)	3.139345 (0.013750)
100		11.937100 (2.544100)	1.215614 (0.004222)
150		1.724300 (0.565700)	0.990972 (0.003156)
250		0.914900 (0.187100)	1.022943 (0.002731)
500		0.6912 (0.178500)	0.845130 (0.001594)

Table 19 shows the estimates of multiplicative error based CES production function model when efficiency parameter γ (1.0), distribution parameter δ (0.6), the substitution parameter τ (0.5), and scale parameter ν (1.1) with the sample size of 50-500 for both classical and Bayesian methods with the Law of diminishing marginal returns to both Labour and Capital.. For GNM, the estimates of $(\gamma$ $=1.0)$ are unreliable, $(\delta$ $=0.6)$ are not close to the true value, $(\tau$ $=0.5)$ obtained varied out of the range with no meaningful standard errors, for example with sample size of 250, the estimate is 67.437800 with standard error 2221.762600and the estimates of $(\nu$ $=1.1)$ for sample size of 150-500 are significantly unconvincing. The estimates under Bayesian approach, $(\gamma$ $=1.0)$ are reliable, $(\delta$ $=0.6)$ are close to the true value, the estimates of $(\tau$ $=0.5)$ obtained for sample sizes of 50-150 (0.954027, 1.069045 and 0.913035) respectively varied out of the range while for sample of 250 and 500 are within the range(0.119093 and 0.468569) respectively with meaningful standard errors, for example with sample size of 250, the estimate is 67.437800 with standard error 2221.762600 and the estimates of $(\nu$ $=1.1)$ are significantly plausible except for the sample size of 50. Findings in this table reveled that Bayesian approach is preferable.

Table 20: The Multiplicative Error Based CES Production Function for Effect of Change in Inputs $\tau(0)$

Sample size	True value	GNM	Bayesian
50		2.323400 (0.943100)	1.268762 (0.006313)
100		2.264000 (0.411200)	1.048194 (0.004128)
150		1.650000 (0.342000)	1.488793 (0.003292)
250	γ (1.0)	1.522300 (0.199400)	0.989757 (0.002518)
500		1.492000 (0.153600)	0.994861 (0.001705)
50		0.341100 (0.437800)	0.776390 (0.007535)
100		0.000000 (0.000000)	1.449393 (0.006217)
150		4.789000 (4.447000)	1.0562638 (0.004236)
250	δ (0.6)	0.999900 (0.037200)	0.784247 (0.002220)
500		1.000000 (0.000200)	0.972400 (0.001715)
50		0.112900 (4.348800)	0.316000 (0.005696)
100		-1.972000 (1.160000)	0.212574 (0.002313)
150		0.000900 (0.097900)	0.247382 (0.001739)
250	τ (0)	**-5.831900(190.593100)**	**0.098531 (0.001117)**
500		-2.471000 (2.562000)	0.002003 (0.000775)
50		0.624800 (0.710000)	0.573869 (0.010370)
100		0.645000 (0.339900)	0.106641 (0.003284)
150		-0.002500 (0.230700)	0.727136 (0.003100)
250	v (1.1)	-0.098000 (0.134800)	0.073202 (0.003098)
500		-2.471000 (2.562000)	-0.004895 (0.001724)

Table 20 shows the estimates of multiplicative error based CES production function model when efficiency parameter γ (1.0), distribution parameter δ (0.6), the substitution parameter τ (0), and scale parameter ν (1.1) with the sample size of 50-500 for both classical and Bayesian methods with the Law of diminishing marginal returns to both Labour and Capital.. The estimates of GNM, $(\gamma =1.0)$ are implausible, the estimates of $(\delta =0.6)$ for sample sizes of 50 and 100 are 0.341100 and 0.0000000 respectively are reliable while for sample sizes of 150-500 are not are close to the true value. The estimates of $(\tau =0)$ obtained are not within the specified range except in the sample size of 150 where its estimate is 0.000900 with the standard error of 0.097900 and the estimates of $(\nu =1.1)$ for sample size of 50-100 are plausible while 150-500 are not otherwise. The estimates under Bayesian approach for $(\gamma =1.0)$ are not reliable for sample sizes of 50 and 150 while the estimates, 1.048194, 0.989757 and 0.994861 for sample 100, 250 and 500 respectively are close to the true value and reliable. The estimates of $(\delta =0.6)$ are close to the true value except when sample sizes are 100 and 150; the estimates of $(\tau =0)$ obtained for all the sample sizes from 50- 500 (0.316000, 0.212574, 0.247382,0.098531 and 0.002003) respectively fall within the range with meaningful standard errors and the estimates of $(\nu =1.1)$ are reliable. This table shows that the estimates under Bayesian approach behaved better than that of classical approach.

Table 21: The Multiplicative Error Based CES Production Function for Effect of Change in Inputs $\tau(-0.5)$

Sample size	True value	GNM	Bayesian
50	γ (1.0)	3.690500 (2.221700)	1.282120 (0.006063)
100		9.431790 (2.019800)	0.888796 (0.004667)
150		1.943000 (0.467900)	1.191310 (0.003579)
250		1.681293 (0.239674)	0.948771 (0.002612)
500		1.312780 (0.132500)	0.065123 (0.001784)
50	δ (0.6)	0.834500 (0.356500)	0.623575 (0.006197)
100		0.051090 (0.068640)	0.579199 (0.004923)
150		0.055000 (6.206000)	0.608084 (0.003849)
250		0.4088175(0.117386)	0.444443 (0.002898)
500		0.506490 (0.088090)	0.594937 (0.001922)
50	τ (-0.5)	2.587800 (6.459600)	-0.447545 (0.003087)
100		5.432950 (4.886390)	-0.399159 (0.001634)
150		5.366000 (2.218000)	-0.039392 (0.001803)
250		**0.004485 (0.849646)**	**-0.316245 (0.001215)**
500		-0.373520 (0.462340)	-0.726498 (0.000621)
50	ν (1.1)	2.624300 (1.658900)	1.211103 (0.008522)
100		11.031520 (2.354680)	0.522342 (0.004759)
150		0.964100 (0.298500)	1.402227 (0.003800)
250		1.051127 (0.298681)	1.595491 (0.002995)
500		0.751850 (0.181950)	0.137869 (0.001855)

Table 21 shows the estimates of multiplicative error based CES production function model for classical and Bayesian approaches when efficiency parameter γ (1.0), distribution parameter δ (0.6), the substitution parameter τ (-0.5), and scale parameter ν (1.1) with the sample size of 50-500 for both classical and Bayesian methods with the Law of diminishing marginal returns to both Labour and Capital.. For GNM, the estimates of $(\gamma =1.0)$ are unreliable, $(\delta =0.6)$ are close to the true value except when the sample size is 50, $(\tau =0.5)$ obtained for 50 – 250 varied out of tolerable range except when the sample size is 500. The estimates of $(\nu =1.1)$ for sample size of 150-500 are significantly plausible except for sample sizes of 50 1nd 100. The estimates under Bayesian approach, $(\gamma =1.0)$ are reliable, $(\delta =0.6)$ are close to the true value, the estimates of $(\tau = -0.5)$ are within the tolerable range with meaningful standard errors, and the estimates of $(\nu =1.1)$ are significantly implausible except for the sample size of 100 and 500.In this table, the standard error decreased more consistently as sample size increased under Bayesian approach than that of classical approach. This consequentially makes the Bayesian approach preferable.

Table 22: Standard Errors of the CES production functionwhen τ= -0.5 with sample sizes (N); 50, 100, 150, 250 and 500

Sample size	GNM				Bayesian			
	γ = 1.0	δ = 0.6	τ= -0.5	ν= 1.1	γ = 1.0	δ = 0.6	τ= -0.5	ν= 1.1
50	2.221700	0.356500	6.459600	1.658900	0.006063	0.006197	0.003087	0.008522
100	2.019800	0.068640	4.886390	2.354680	0.004667	0.004923	0.001634	0.004759
150	0.467900	6.206000	2.218000	0.298500	0.003579	0.003849	0.001803	0.003800
250	0.239674	0.117386	0.849646	0.298681	0.002612	0.002898	0.001215	0.002995
500	0.132500	0.088100	0.462300	0.182000	0.001784	0.001922	0.000621	0.001855

Table 22 shows the standard errors of the multiplicative error based CES production functions for (τ) fixed at a given value in classical and Bayesian approaches for the sample sizes of 50, 100, 150, 250, 500.For GNM, it was revealed that as sample sizes increased the standard errors for the substitution parameter (τ) and efficiency parameter (γ) consistently decreased, while the standard errors of distribution parameter (δ) and that of scale parameter (ν)do not decrease with increase in sample size resulting to inconsistent estimates. For Bayesian approach, the standard errors of the efficiency parameter (γ), distribution parameter (δ), substitution parameter (τ) and scale parameters (ν) are all decreased as sample size increased which agreed with the sufficient condition of consistency for nonlinear estimate for suitable initial values; that standard Errors (S.E)tend to zero as sample sizes increases indefinitely. This has been proven by Chien (1981) and Debasis (1993) that for a suitable choice of initial values, the nonlinear least square produces consistent estimates under the assumption of the normal error.

Table 23: Standard Errors of the CES production function when $\tau = 0$ with sample sizes (N); 50, 100, 150, 250 and 500

Sample size	GNM				Bayesian			
	γ = 1.0	δ = 0.6	τ = 0	ν = 1.1	γ = 1.0	δ = 0.6	τ = 0	ν = 1.1
50	0.943100	0.437800	4.348800	0.710000	0.006313	0.007535	0.005696	0.010370
100	0.411200	0.000000	1.160000	0.339900	0.004128	0.006217	0.002313	0.003284
150	0.342000	4.447000	0.097900	0.230700	0.003292	0.004236	0.001739	0.003100
250	0.199400	0.037200	190.593100	0.134800	0.002518	0.002220	0.001117	0.003098
500	0.153600	0.000200	2.562000	2.562000	0.001705	0.001715	0.000775	0.001724

Table 23 shows the standard errors of the multiplicative error based CES production functions for (τ) fixed at a given value in classical and Bayesian approaches for the sample sizes of 50, 100, 150, 250, 500. For GNM, the standard errors of efficiency parameter (γ) decreased as sample size increased, the standard errors of distribution parameter (δ) decreased as sample size increased from 50,150-500 while sample size of 100 has zero standard error. For Bayesian approach, the standard errors of the efficiency parameter (γ), distribution parameter (δ), substitution parameter (τ) and scale parameters (ν) are all decreased as sample size increased which agreed with the sufficient condition of consistency for nonlinear estimate for suitable initial values; that standard Errors (S.E) tend to zero as sample sizes increases indefinitely.

Table 24: Standard Errors of the CES production function when $\tau = 0.5$ with sample sizes (N); 50, 100, 150, 250 and 500

Sample size	GNM				Bayesian			
	γ = 1.0	δ = 0.6	τ = 0.5	ν = 1.1	γ = 1.0	δ = 0.6	τ = 0.5	ν = 1.1
50	2.286300	0.344900	6.297900	1.699000	0.008985	0.012560	0.005785	0.013750
100	2.180800	0.069700	4.963900	2.544100	0.004301	0.007983	0.003074	0.004222
150	0.463700	0.129000	1.122100	0.565700	0.003695	0.006410	0.001808	0.003156
250	0.312100	7.197800	2221.762600	0.187100	0.002506	0.002597	0.001380	0.002731
500	0.117500	0.277300	5.476700	0.178500	0.001659	0.001789	0.000817	0.001594

Table 24 shows the standard errors of the multiplicative error based CES production functions for (τ) at a given value in classical and Bayesian approaches with the Law of diminishing marginal returns to both Labour and Capital, for the sample sizes of 50, 100, 150, 250 and 500. For GNM, findings showed that as sample sizes increased the standard errors for efficiency parameter (γ) and scale parameter (ν) consistently decreased, the standard errors of distribution parameter (δ) decreases as sample size increases except for sample size 250 and for the substitution parameter (τ) the standard errors has no meaningful information when the sample size increased. For Bayesian approach, the standard errors of the efficiency parameter (γ), distribution parameter (δ), substitution parameter (τ) and scale parameters (ν) are decreased as sample size increased. This implies that Bayesian method is a suitable estimation method of CES production functions because it produced minimal standard errors compared to that of classical method.

Table 25 Estimates of CES production Function when $\tau > 0$ then $\Rightarrow$ $\sigma < 1$; N=50, 100, 150, 250 and 500

$\tau = 0.5$	**GNM**	**Posterior mean**
N= 50	0.277631	0.511764
N=100	0.157260	0.483315
N=150	1.033165	0.522730
N=250	**0.014612**	**0.893581**
N=500	0.254693	0.680935

Table 25 summarizes the estimates obtained for all the two methods under CES production function when $\tau > 0$(positive) for any valve of 'v' (Elasticity of scale parameter) greater than 1 i.e $v > 1$, then with the continual growth in Labour-Capital, the Elasticity of substitution (σ) becomes positively less than 1 ($\sigma < 1$) for large sample size. This table shows the estimates of GNM are within the range of elasticity of substitution except when N=150. The estimates under Bayesian (posterior mean) are all within the range of elasticity of substitution (σ). This implies that the estimates under Bayesian method do behaved well while the estimates of GNM does not with the Law of diminishing marginal returns to both Labour and Capital.

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Table 26 Estimates of CES production Function when $\tau < 0$ then $\Rightarrow \sigma > 1$; N=50, 100, 150, 250 and 500

$\tau = -0.5$	**GNM**	**Posterior mean**
N= 50	0.278722	1.810102
N=100	0.155448	1.664334
N=150	0.157085	1.041007
N=250	**0.995520**	**1.462512**
N=500	1.596424	3.656280

Table 26 shows the estimates for the CES production function under classical and Bayesian approaches when $\tau < 0$(negative) for any value of $'v'$(Elasticity of scale parameter) greater than 1, i.e. $v > 1$, the Elasticity of substitution exceed 1 (σ>1), measured the varying factors substituted. In the finding, as samples sized increase, the estimates of GNM are not within the range for all sample sizes except 500 while that of Bayesian falled within the range of elasticity of substitution with the Law of diminishing marginal returns to both Labour and Capital.

Table 27. Estimates of CES production Function when $\tau=0$ then $\Rightarrow \sigma=1$; N=50, 100, 150, 250 and 500

$\tau=0$	**GNM**	**Posterior mean**
N= 50	0.898553	0.759878
N=100	-1.028810	0.824692
N=150	0.999101	0.801679
N=250	**-0.206960**	**0.910307**
N=500	-0.679810	0.998001

Table 27 contains the estimates of classical and Bayesian approaches for the CES production function with the Law of diminishing marginal returns to both Labour and Capital, when $\tau=0$, for any value of $'v'$(Elasticity of scale parameter) greater than 1($v>1$) with the constant Elasticity of substitution($\sigma=1$). It was showed that as samples size increased, the estimates of classical approach do not behave well. The estimates of Bayesian approaches do behave well. In addition, the estimates of Bayesian method approximately equal one.

CHAPTER FIVE

Summary of Findings, Conclusion and Recommendation

The study estimated the parameters of nonlinear C-D production functions with multiplicative and additive error based specification using both the classical and Bayesian approaches. The behaviours of parameters set $\beta = (\beta_0, \beta_1, \beta_2)$ under three scenarios of C-D as sample size increased were investigated with respect to standard errors of the parameters estimates. Table 1-5 show the estimates of C-D for constant returns to scale with additive and multiplicative error based model in classical and Bayesian approaches for all sample sizes. Thus, from the findings, the estimates were closer to the true values with minimal posterior standard errors under Bayesian approach. The estimates of Bayesian estimation behaved well. In Table 6-10, the estimates of increasing returns to scale of C-D production function with additive and multiplicative error based model in classical and Bayesian for sample sizes were considered. The estimates of classical approach for C-D production function with additive error specification were closer compared to that multiplicative error. However, Bayesian approach produced the posterior means that are closer to the true values with minimal standard errors for both Additive and Multiplicative model than classical approach as sample size increased. The information in tables 11-15 revealed that the estimates of C-D with additive and multiplicative error based model in classical and Bayesian for decreasing return to scale for all sample sizes. The estimates of classical approach for C-D production function with additive error were close to true values and behaved well compared to that multiplicative error. Moreover, the Posterior estimates with minimal standard errors were produced in Bayesian approach for Additive and Multiplicative model are closer to the true values and behaved better. Thus, Bayesian estimator performed better than Classical estimator in estimating the Decreasing Returns to scale for C-D production function.

Table 16-18 give the summary of the standard errors of all the additive and multiplicative errors –based model in classical and Bayesian for the three scenarios of cobb- Douglas production functions. The results showed that standard errors of the estimators decreased consistently as sample size increased for both classical and Bayesian approaches. In addition to findings in classical approach, the estimates for additive error based model reduced consistently with meaningful standard errors compared that of multiplicative error based model. However, the Bayesian approach was preferred because it has minimal standard errors.

Table 19-21 show the estimates of the parameters of nonlinear CES production functions with multiplicative error based specification using both the classical and Bayesian approaches. The effects of varying substitution parameter, $\tau = -0.5, 0 \; and \; 0.5$ on other parameters (γ, δ, ν) in CES were observed with respect to standard errors of the parameters as sample size increases with independent normal gamma prior. The priors used are 0.001, 0.005 and 0.01 under the various sample sizes of study with the aid of metropolis -Hasting -Within Gibbs algorithm to obtain the posterior estimates as recorded in the tables in previous chapter. Techniques of Monte Carlo Simulation are used in the study. From the findings in table 19-21, the estimates obtained under classical approach were not close to the true values while the estimates for Bayesian for efficiency parameter (γ), substitution parameter (τ) and scale parameter (ν) were close to the true values and significantly plausible. However, the estimates for the substitution parameter τ (0.5) are insignificantly plausible, with substitution parameter τ $(0 \text{ and } \text{-}0.5)$, the estimates were within the tolerable range with meaningful standard errors with the Law of diminishing marginal returns to both Labour and Capital. Table 22-24 contain the information about the standard errors of CES production function at considered values of substitution parameter in classical and Bayesian approaches. Findings revealed that as sample sizes increased the standard errors for efficiency parameter (γ) consistently decreased while that of distribution parameter (δ) were inconsistent in table 22-24. The standard errors of substitution parameter are inconsistent in table 23 and 24; and the standard errors of scale parameter were not also consistent in table 23.The standard errors of efficiency parameter

(γ),distribution parameter (δ),substitution parameter (τ) and scale parameter (ν) consistently decreased as sample size increased. Hence, the standard errors of the Bayesian approach decreased for all parameters. In effect, Bayesian produced better results which make it appropriate in handling nonlinear CES production functions. In Table 25-27, the findings revealed that the estimates of elasticity of substitution obtained did not behave well. Also, the estimates when the elasticity of substitutions were lesser and greater than one were within the range except when the sample sizes are 150 and 500 respectively in classical approach. For Bayesian approach, the estimates of elasticity of substitution obtained for all sample sizes did behave well when it is equal to one, lesser and greater than one with the Law of diminishing marginal returns to both Labour and Capital.

5.1 Findings and Conclusions

This work revealed interesting results which are useful in empirical study in both practical application and methodological relevance. The work showed that the ways the errors were specified in the model determined the behaviour of the parameters of CD and the substitution parameter of CES also determined the behaviour of its parameters. The Table 1-15 above showed the estimates obtained using the Cobb-Douglas production function with additive error and multiplicative error term for all the three scenarios.

A Metropolis -Hasting - Within Gibbs Algorithm was used with the independent Normal-Gamma prior to obtain the posterior estimates as recorded above in Table 1-15 using the additive error model and multiplicative error model. It was discovered that at all sample sizes the multiplicative error model behaved better than the additive error model in Bayesian approach, also the NSE in Cobb-Douglas production function with multiplicative produced minimum values all through their sample sizes; 50, 100, 150, 250 and 500. Furthermore, the nuisance parameter β_0 showed a fluctuated and unsteady behaviour by producing values close to the true parameters and values that are far from the true values, aside this, every other estimate was not too far from the true values for both the classical approach and the Bayesian approach. In Table 16- 18 the standard error of the

GNM and the Standard Error of the posterior decreased consistently as sample size increased. The standard errors of the estimates obtained in Bayesian approach outperformed the classical approach under the three scenarios of Cobb Douglas production functions.

Findings from Table 19-24 in previous chapter showed that for given $\gamma = 1.0$, $\delta = 0.6$, $\nu = 1.1$ and $\tau = -0.5,\ 0$ and 0.5 for all sample sizes of 50, 100, 150, 250 and 500 in CES production function, the estimates of Bayesian behaved well by giving minimal standard errors compared to that of GNM. On the contrary, the estimates of GNM showed unsteady behaviour to the parameter values. Table 25-27 revealed that for all sample sizes the estimates of elasticity of substitution σ less than one, equal to one and greater one were within the range specified and well behaved in Bayesian. However, estimates of GNM revealed unsteady behaviour by giving values that are not within the specified range. From literatures, the substitution parameter (τ) is the most challenging parameter of the CES production function that often gives ambigious values. Iyaniwura (1974) confirmed that substitution parameter produced ambigious values in CES production function using classical approach. Similarly, we encountered same challenge when using the classical approach; the reason for pegging (omitting some values of τ) some values of substitution parameters but the Bayesian approach took care of this limitation; which makes the Bayesian Approach more suitable approach.

5.2 Conclusion

The objectives of this work have been achieved by investigating the behaviour of the parameters of Cobb Douglas based on the error specification and CES production with the substitution parameter not equal to one. From the two models considered in C-D, the C-D based Additive Error Model performed better than its Multiplicative Error model in Classical approach; thus, the estimates of additive error-based behaved better than multiplicative error- based Cobb- Douglas production model. Bayesian estimates for multiplicative error- based model behaved better than additive error- based model. This consequentially makes Bayesian more preferable than classical approach. The derived

Bayesian estimators of nonlinear production functions in the both C-D with additive and multiplicative errors models under the Decreasing, constant and Increasing returns to scale and CES under varying substitution parameter validates the statistical inference. In effect, this approach outperforms the existing classical approach which sufficiently depends on the observed values at hand.

5.3 Research Contribution

This study has particularly contributed to knowledge in following ways;

i. This work extended koop (2003) by investigating the effect of substitution parameter when it assumes different values in Bayesian and Classical frameworks as against the substitution parameter to be unity proposed by Koop (2003).

ii. We investigated the behaviour of the parameters on the proposed Bayesian estimation of the Cobb-Douglas production function with multiplicative and additive based -error under three specifications of returns to scale parameters against the multiplicative based – error model by Koop (2003).

iii. We extended the nonlinear regression model of Koop (2003) by incorporating a form of complex prior known as the independent Normal-Gamma Prior for parameters (β, h) with the Likelihood to obtain a non-analytical (intractable) posterior, whose posterior estimates were obtained using a Markov Chain Monte Carlo technique of a Metropolis within Gibbs algorithm as against the Independent and Random work Metropolis by Koop (2003).

iv. The work of Martinez (2012) was extended by using Kmenta approximation (1967) of CES in the neighbourhood of substitution parameters, $\tau \to 0$to form the distribution of the likelihood as against ordinary least squares (OLS) and two stage least squares (2sls) Martinez (2012).

v. An R code for Metropolis within Gibbs Algorithm was developed to obtain the conditional densities and executing the Cobb-Douglas and CES production functions and showed the evidence of convergence.

5.4 Further Study

1. There could be some violations of basic assumption like multicollinearity and heteroscedasticity in nonlinear production function using Bayesian approach
2. Since standard linear models used to estimate nonlinear production function are not susceptible to the curse of dimensionality, the use of non-parametric approach could be considered to handle this problem.
3. More importantly, an independent normal – gamma prior in this work could be extended to hierarchical prior.

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6.0 Appendix 1

6.1 R. Code

THE R CODES USED FOR THIS STUDY

##

THE BEHAVIOUR OF PARAMETERS OF NONLINEAR PRODUCTION FUNCTIONS USING BAYESIAN AND CLASSICAL APPROACHES

##

FOR COBB DOUGLAS ADDITIVE ERROR MODEL

##

The Algorithm below gives the set of data simulated using sample size of 50, the data analysis for the other sample sizes can be obtained by just changing the sample sizes to the desired one using the R commands.

```
getwd()

set.seed(01)

N = 50; B0 = 15; B1 = 0.85; B2 = 0.15

x1 = runif(N,0,1) head(x1)

x2 = runif(N,0,1) y1 = Y; x1 = k; x2 = L

#N = length(y1);B0 = 15; B1 = 0.85; B2 = 0.15

set.seed(123)

U = rnorm(N,0,1)

v = U/(B0*x1^B1*x2^B2)
```

```
y1 = B0*x1^B1*x2^B2*(1+v)

head(y1) cobbADD = data.frame(y1, x1, x2)

 write.csv(cobbADD, "cobbadd.csv")

X = cbind(1, x1, x2) X = as.matrix(X)

head(X) #cobbADD = data.frame(y,x1,x2,e)

write.csv(cobbADD, "cobbadd.csv", row.names=F)

##############################################################################

 checking the reliability of the fixed beta-values -valuesols = lm(y~X)ols

##############################################################################

 THE NONLINEAR REGRESSION MODEL

##############################################################################

 taking log of bothsides of the Cobb-Douglass function

ln(y) = B0+B1*x1+B2*x2+e  where B0 =ln(alpha)

Y = log(y1) Y[is.na(Y)]<-0

Y = as.matrix(Y) head(Y)

X1 = log(x1) head(X1)

X2 = log(x2) head(X2)

X =cbind(1,X1,X2)X = as.matrix(X)head(X)

OLS = lm(Y~X) summary(OLS)

Ols = glm(Y~ X1+X2)

summary(Ols)
```

b0 = exp(coef(OLS)[1]) b0V

xpx=t(X)(X)   x'x xpxinv=solve(xpx) (x'x)^-1

B=(xpxinv)%%(xpy)   B=(x'x)^-1*(x'y)

##

LIKELIHOOD FUNCTION (multivariate normal density)

###

P(y/B,h) = (h^N/2)/(2pi)^N/2{exp[-h/2(y-XB)'(y-XB)]}

y/B,h ~ N(XB,h^-1), where, h=1/sigma^2 (h is precision)# k = 3

#v = N-k   Y=XB

SSE=t((Y-xb))%*%(Y-xb)   e'e=(y-XB)'*(y-XB)

ssqr=SSE/v   ssqr=(y-XB)%%(y-XB)/v

ssqr h = ssqrinv=%*%(ssqr)^-1

##

THE PRIOR (Non-INFORMATIVE PRIOR)

##

P(B,h)=1/h it follows a normal distribution

P(B,h) = h0^N/2*{exp[-h0/2(B-Bpri)'(B-Bpri)]} rbind(0.001,0.005,0.01)

Bprior =%*% as.matrix(Bprior)
Vprior=matrix(c(500.40^2,0,0,0,90.97^2,0,0,0,499.097^2),nrow=3,ncol=3,byrow=T)

h ~ G(ssqrinvpri,vpri), where, ssqrinvpri=s^-2 ssqrinvprior=1/(1000000)^2

ssqrinvprior ssqrprior vprior=0.3 1% of N noninformative prior

##

THE POSTERIOR (NON-ANALYTICAL FORM)

##

p(B,h/y) = p(B,h)*(h^N/2)/(2pi)^N/2{exp[-h/2(y-XB)'(y-XB)]}.

P(B/y,h)=exp(drop(-0.5)*(B-Bpri)(solve(Vpri))%*%(B-Bpri))

Vpos=Vpos=(Vpri^h(xpx)) Vpos

Bpos = Vpos%*%((solve(Vprior)%*%Bprior)(drop(h) pos=Vpos(Vpri^-1+h(xpy))

Bpos

h/y,B ~ G(ssqrinvpos)

vpos=vprior %*%vpos(small v posterior)

vpos vpos = sqrt(vpos)

vpos ssqrpos=(SSE+(vprior*%ssqrprior))

ssqrpos ssqrinvpos=solve%(ssqrpos)% ssqrinvpos

##

METROPOLIS-HASTINGS ALGORITHM

###

Now our target is the posterior density of B and h.We have to use the log of the posterior instead of the actual posterior for computational reasons.

###

This is the actual Metropolis algorithm

##

log-denisty of a multivariate normal for B/h~N(Bpos,Vpos)

return((-(1/2)) * (t(B - Bpos)%*%(solve(Vpos))%*%(B - Bpos))) }else{ return(-Inf) }}log.posterior2 = function(Y, xb, h, vprior, ssqrprior, vpos){ N = 50 vpos = N+vprior

return(((N+vprior-2)/2)* log(h) + (-(h/2) * t(Y-xb)%*%(Y-xb) +vprior*ssqrprior)) }else{ return(-Inf)}.

mcmc = 10000 sampled.beta0 = numeric(mcmc)

sampled.beta1 = numeric(mcmc)sampled.beta2 = numeric(mcmc)

sampled.h = numeric(mcmc) setting starting values

current.sample = rbind(5,10,23) current.sample = as.matrix(current.sample)

current.sample current.h = 10

initialize the loop and run

sampled.beta0[1] = current.sample[1,]

sampled.beta1[1] = current.sample[2,]

sampled.beta2[1] = current.sample[3,]

sampled.h[1] = current.h

##

The following loop will execute the Metropolis Algorithm

###

for(i in 2:mcmc){ the proposal density function for B and h will be done separately. It's a normal density centered at the old value

##

Library (MASS)

##

Vpos = solve(solve(Vprior)+(drop(current.h)*xpx)) VposB%*((solve(Vprior)%*%Bprior)+(dropB(current.h)*xpy)) #Bpos=Vpos(Vpri^-1+h(xpy))

x.star = mvrnorm(1, Bpos, Vpos) x.star = as.matrix(x.star) x.star This line computes the acceptance probability. target(old.sample) is the function value at the old point and target(x.star) is the function value at the proposed point

alpha2 = log.posterior(Bpos = x.star, Vpos = Vpos, B = B) log.posterior(Bpos current.sample, Vpos = Vpos, B = B) if (log(runif(1)) < alpha2){current.sample = x.star } sampled.beta0[i] = current.sample[1,] sampled.beta1[i] = current.sample[2,] sampled.beta2[i] = current.sample[3,]

for h vpos = N+vprior

ssqrpos=(SSEQ+(vprior*ssqrprior))/(vpos)

ssqrinvpos %solve(ssqrpos)

x.star2 = rgamma(1, %%ssqrinvpos, %%vpos) x.star2

alpha2B = log.posterior2(h = x.star2, vpos = vpos, Y=Y, xb=xb, vprior=vprior, ssqrprior=ssqrprior)-log.posterior2(h = current.h, vpos = vpos, Y=Y, xb=xb, vprior=vprior, ssqrprior=ssqrprior) if (logB(runif(1)) < alpha2){ current.h = x.star2 }

sampled.h[i]plot(sampled.beta0, type = 'l', main = "Trace Plot")

plot(sampled.beta1, type = 'l', main = "Trace Plot")

plot(sampled.beta2, type = 'l', main = "Trace Plot")

plot(sampled.h, type = 'l', main = "Trace Plot")mean of the samples before burn-in

```
mean(sampled.beta0meanSampled.betamean(sampled.beta2)meansampled.h)

summary(sampled.beta0)summary(sampled.beta1)summary(sampled.beta2)

### BURN-IN

final.beta0 = sampled.beta0[-(1:1000)]

mean(final.beta0)plot(final.beta0, type = 'l', main = "Trace Plot")

#abline(plot(final.beta0, type = 'l', main = "Trace Plot"))

hist(final.beta0, prob=T) lines(density(final.beta0), col="red" )

final.beta1 = sampled.beta1[-(1:1000)]

mean(final.beta1)

hist(final.beta1, prob=T) lines(density(final.beta1), col="purple" )

plot(density(final.beta1), col="purple" final.beta2 = sampled.beta2[-(1:1000)]

mean(final.beta2)hist(final.beta2, prob=T)

lines(density(final.beta2), col="darkblue" )plot(density(final.beta2), col="darkblue" final.h
= sampled.h[-(1:1000)]mean(final.h)st(final.h, prob=T) ines(density(final.h), col="red" )

plot(density(final.h), col="red" )mean(final.beta0)mean(final.beta1)

mean(final.beta2)mean(final.h)

## USING CODA

library('coda')b0.mcmc=mcmc(final.beta0) summary(b0.mcmc)

plot(b0.mcmc, col="pink")

title('beta0',xlab='mcmc', ylab='b0.mcmc')

(final.beta1)summary(b1.mcmc) plot(b1.mcmc, col="cyan")
```

```
title(',xlab='mcmc', ylab='b1.mcmc')

b2.mcmc=mcmc(final.beta2)

summary(b2.mcmc)plot(b2.mcmc, col="purple")

title('beta2',xlab='mcmc', ylab='b2.mcmc')

h.mcmc=mcmc(final.h)                summary(h.mcmc)

plot(h.mcmc, col="red")

title('h',xlab='mcmc', ylab='h.mcmc')

#library('coda')

### testing

#print(fit, digits_summary=3, pars=c('final.beta0','final.h'),

#     probs = c(.025, .5, .975))

## GEWEKE'S CONVERGENCE DIAGNOSTICS (GEWEKE'S CD)

## frac1 is the fraction of d 1st-set of est of B after burnin i.e 1000/10000

## frac2 is the fraction of d last-set of est of B after burnin i.e 9000/10000

## DECISION RULE: if CD<1.96, then convergence of d--

#   --MCMC algorithm has occured for all the parameters

library(coda)

geweke.diag(final.beta0, frac1=0.15, frac2=0.85)

geweke.diag(final.beta1, frac1=0.15, frac2=0.85)

geweke.diag(final.beta2, frac1=0.15, frac2=0.85)

#geweke.diag(final.h, frac1=0.1 frac2=0.9)
```

##

INVESTIGATION OF THE BEHAVIOUR OF PARAMETERS OF NONLINEAR PRODUCTION FUNCTIONS USING BAYESIAN AND CLASSICAL APPROACHES

##

FOR COBB DOUGLAS MULTIPLICATION ERROR MODEL

##

The Algorithm below gives the set of data simulated using sample size of 50, the data analysis for the other sample sizes can be obtained by just changing the sample sizes to the desired one using the R commands.

```
getwd()

N = 50; B0 = 15; B1 = 0.85; B2 = 0.15

set.seed(01)

x1 = runif(N,0,1) head(x1)

set.seed(11)x2 = runif(N,0,1)

set.seed(123)

e = rnorm(N,0,1)head(e)

y = B0*x1^B1*x2^B2*exp(e) head(y)

cobbMULT = data.frame(y,x1,x2)

write.csv(cobbMULT, "cobbmult.csv")

X = cbind(1, x1, x2)X = as.matrix(X)

head(X)
```

##

checking the reliability of the fixed beta-values

##

ols = lm(y~X)

ols

##

using non- linear directly

##

library(minpack.lm)

nl = nls(y~B0*x1^B1*x2^B2, start = list(B0=15, B1=0.65, B2=0.15))

nl take exponential of B0summary(nl)

##

THE NONLINEAR REGRESSION MODEL FOR COBB DOUGLAS MULTIPLICATIVE ERROR MODEL

###

taking log of bothsides of the Cobb-Douglass function

ln(y) = B0+B1*x1+B2*x2+e # where B0 =ln(alpha)

Y = log(y) Y = as.matrix(Y)head(Y)

X1 = log(x1)#head(X1)

X2 = log(x2)

head(X2)X =cbind(1,X1,X2)

X = as.matrix(X)

head(X)cobbLOGmult = data.frame(Y, X1, X2)

write.csv(cobbLOGmult, "cobbLOGmult.csv")YOLS = lm(Y~X)

OLS summary(OLS)

Ols = glm(Y~ X1+X2)

summary(Ols)

b0 = exp(coef(OLS)[1])b0xpx=t(X)%*%(X)   x'x

xpxinv=solve(xpx)   (x'x)^-1

xpy=t(X)%*%(Y)   x'yB=(xpxinv)%*%(xpy)   # B=(x'x)^-1*(x'y)

##

LIKELIHOOD FUNCTION (multivariate normal density)

##

P(y/B,h) = (h^N/2)/(2pi)^N/2{exp[-h/2(y-XB)'(y-XB)]} y/B,h ~ N(XB,h^-1), where, h=1/sigma^2 (h is precision)k = 3

v = N-kxb=(X)%*%(B) %%Y=XB

SSE=t((Y-xb))%(Y-xb)   e'e=(y-XB)'*(y-XB)

ssqr=SSE/v ssqr=(y-XB)*(y-XB)/v

ssqrh = ssqrinv=(ssqr)^-1%%h

##

THE PRIOR (Non-INFORMATIVE PRIOR)

##

P(B,h)=1/h ("=" stands for proportionality sign here)

```
## it follows a normal distribution

P(B,h) = h0^N/2*{exp[-h0/2(B-Bpri)'(B-Bpri)]} rbind(0.001,0.005,0.01)

Bprior = as.matrix(Bprior)

Vprior=matrix(c(500.40^2,0,0,0,90.97^2,0,0,0,499.097^2),nrow=3,ncol=3,byrow=T)

h ~ G(ssqrinvpri,vpri), where, ssqrinvpri=s^-2 ssqrinvprior=1/(1000000)^2

ssqrinvprior ssqrprior vprior=0.3   # 1% of N

 noninformative prior

##########################################################################

THE POSTERIOR (NON-ANALYTICAL FORM)

##########################################################################

 p(B,h/y) = p(B,h)*(h^N/2)/(2pi)^N/2{exp[-h/2(y-XB)'(y-XB)]}.

P(B/y,h)=exp(drop(-0.5)*(B-Bpri)(solve(Vpri))%*%(B-Bpri))

Vpos=Vpos=(Vpri^-1+h(xpx)) Vpos

Bpos = Vpos%*%((solve(Vprior)%*%Bprior)+(drop(h) pos=Vpos(Vpri^-1+h(xpy))

Bpos

h/y,B ~ G(ssqrinvpos,vpos)

vpos=N+vprior        #vpos(small v posterior)

vpos vpos = sqrt(vpos)

vpos ssqrpos=(SSE+(vprior*ssqrprior))

ssqrpos ssqrinvpos=solve(ssqrpos) ssqrinvpos

##############################################################
```

METROPOLIS-HASTINGS ALGORITHM

##

Now our target is the posterior density of B and h.We have to use the log of the posterior instead of the actual posterior for computational reasons.

###

This is the actual Metropolis algorithm

###

log-denisty of a multivariate normal for B/h~N(Bpos,Vpos)

return((-(1/2)) * (t(B - Bpos)%*%(solve(Vpos))%*%(B - Bpos))) }else{ return(-Inf) }}log.posterior2 = function(Y, xb, h, vprior, ssqrprior, vpos){ N = 50 vpos = N+vprior

return(((N+vprior-2)/2)* log(h) + (-(h/2) * t(Y-xb)%*%(Y-xb) +vprior*ssqrprior)) }else{ return(-Inf)}.

mcmc = 10000 sampled.beta0 = numeric(mcmc)

sampled.beta1 = numeric(mcmc)sampled.beta2 = numeric(mcmc)

sampled.h = numeric(mcmc) setting starting values

current.sample = rbind(5,10,23) current.sample = as.matrix(current.sample)

current.sample current.h = 10

initialize the loop and run

sampled.beta0[1] = current.sample[1,]

sampled.beta1[1] = current.sample[2,]

sampled.beta2[1] = current.sample[3,]

sampled.h[1] = current.h

##

The following loop will execute the Metropolis Algorithm

###

for(i in 2:mcmc){ the proposal density function for B and h will be done separately. It's a normal density centered at the old value

##

Library (MASS)

##

Vpos = solve(solve(Vprior)+(drop(current.h)*xpx)) VposB%*((solve(Vprior)%*%Bprior)+(dropB(current.h)*xpy)) #Bpos=Vpos(Vpri^-1+h(xpy))

x.star = mvrnorm(1, Bpos, Vpos) x.star = as.matrix(x.star) x.star This line computes the acceptance probability. target(old.sample) is the function value at the old point and target(x.star) is the function value at the proposed point

alpha2 = log.posterior(Bpos = x.star, Vpos = Vpos, B = B) log.posterior(Bpos current.sample, Vpos = Vpos, B = B) if (log(runif(1)) < alpha2){current.sample = x.star } sampled.beta0[i] = current.sample[1,] sampled.beta1[i] = current.sample[2,] sampled.beta2[i] = current.sample[3,]

for h vpos = N+vprior

ssqrpos=(SSEQ+(vprior*ssqrprior))/(vpos)

ssqrinvpos %solve(ssqrpos)

x.star2 = rgamma(1, %%ssqrinvpos, %%vpos) x.star2 alpha2B = log.posterior2(h = x.star2, vpos = vpos, Y=Y, xb=xb, vprior=vprior, ssqrprior=ssqrprior)-log.posterior2(h = current.h, vpos = vpos, Y=Y, xb=xb, vprior=vprior, ssqrprior=ssqrprior) if (logB(runif(1)) < alpha2){ current.h = x.star2 }

sampled.h[i]plot(sampled.beta0, type = 'l', main = "Trace Plot")

plot(sampled.beta1, type = 'l', main = "Trace Plot")

plot(sampled.beta2, type = 'l', main = "Trace Plot")

plot(sampled.h, type = 'l', main = "Trace Plot")mean of the samples before burn-in

mean(sampled.beta0meanSampled.betamean(sampled.beta2)meansampled.h)

summary(sampled.beta0)summary(sampled.beta1)summary(sampled.beta2)

BURN-IN

final.beta0 = sampled.beta0[-(1:1000)]

mean(final.beta0)plot(final.beta0, type = 'l', main = "Trace Plot")

#abline(plot(final.beta0, type = 'l', main = "Trace Plot"))

hist(final.beta0, prob=T) lines(density(final.beta0), col="red")

final.beta1 = sampled.beta1[-(1:1000)]

mean(final.beta1)

hist(final.beta1, prob=T) lines(density(final.beta1), col="purple")

plot(density(final.beta1), col="purple" final.beta2 = sampled.beta2[-(1:1000)]

mean(final.beta2)hist(final.beta2, prob=T)

lines(density(final.beta2), col="darkblue")plot(density(final.beta2), col="darkblue" final.h = sampled.h[-(1:1000)]mean(final.h)st(final.h, prob=T) ines(density(final.h), col="red")

```
plot(density(final.h), col="red" )mean(final.beta0)mean(final.beta1)

mean(final.beta2)mean(final.h)
```

USING CODA

```
library('coda')b0.mcmc=mcmc(final.beta0) summary(b0.mcmc)

plot(b0.mcmc, col="pink")

title('beta0',xlab='mcmc', ylab='b0.mcmc')

(final.beta1)summary(b1.mcmc) plot(b1.mcmc, col="cyan")

title(',xlab='mcmc', ylab='b1.mcmc')

b2.mcmc=mcmc(final.beta2)

summary(b2.mcmc)plot(b2.mcmc, col="purple")

title('beta2',xlab='mcmc', ylab='b2.mcmc')

h.mcmc=mcmc(final.h)            summary(h.mcmc)

plot(h.mcmc, col="red")

title('h',xlab='mcmc', ylab='h.mcmc')

library('coda')

print(fit, digits_summary=3, pars=c('final.beta0','final.h'),
  probs = c(.025, .5, .975))
```

GEWEKE'S CONVERGENCE DIAGNOSTICS (GEWEKE'S CD)

frac1 is the fraction of d 1st-set of est of B after burnin i.e 1000/10000

frac2 is the fraction of d last-set of est of B after burnin i.e 9000/10000

DECISION RULE: if CD<1.96, then convergence of d--

--MCMC algorithm has occured for all the parameters

```
library(coda)

geweke.diag(final.beta0, frac1=0.15, frac2=0.85)

geweke.diag(final.beta1, frac1=0.15, frac2=0.85)

geweke.diag(final.beta2, frac1=0.15, frac2=0.85)

geweke.diag(final.h, frac1=0.1 frac2=0.9)

geweke.diag(final.h, frac1=0.1 frac2=0.9)


################################################################################
```

INVESTIGATION OF THE BEHAVIOUR OF PARAMETERS OF NONLINEAR PRODUCTION FUNCTIONS USING BAYESIAN AND CLASSICAL APPROACHES

##

NONLINEAR REGRESSION MODEL WITH INDEPENDENT NORMAL-GAMMA PRIOR

##

CONSTANT ELASTICITY OF SUBSTITUTION WITH MULTIPLICATIVE ERROR TERMS USING CLASSICAL APPROACH

```
################################################################################

library(micEconCES)

gamma = 1; delta=0.75; rho=0;nu=1.5

N=500
```

```
set.seed(01)

x1 = runif(N,0,1); x2 = runif(N,0,1)

set.seed(431)

u = rnorm(N,0,1)

CES_dat = data.frame(x1,x2)

x = cbind(x1,x2)

##############################################################################

 WITH MULTIPLICATIVE ERROR TERM

##############################################################################

CES_dat$y1  = gamma*(delta*x1^(-rho)+(1-delta)*x2^(-rho))^(-nu/rho)*exp(u)

head(CES_dat)

write.csv(CES_dat, "ces.csv")

############## CES FUNCTION WITH MULTIPLICATIVE ERROR TERM #######

nl_reg = nls(y1~gamma*(delta*x1^(-rho)+(1-delta)*x2^(-rho))^(-phi/rho), data=CES_dat,
start = c(gamma=0.5,delta=0.5,rho=0.25,phi=1))

summary(nlreg)

#####################################################################

library(minpack.lm)

nl_reg_2 = nlsLM(y1~gamma%%(delta*x1^(-rho)+(1-delta)%%*%%x2^(-rho))^(-
phi/rho), data=CESdat, start = c(gamma=0.5,delta=0.5,rho=0.25,phi=1))

summary(nl_reg_2)
```

```
################################################################################
###########   THE KMENTA APPROXIAMTION WITH MULTIPLICATIVE ERROR TERM ###########
kmenta_CES = CSVcesEst(yName="y1",xNames=c("x1","x2"), data=CES_dat,method="Kmenta",=TRUE)
summary(kmenta_CES)
################################################################################
CONSTANT ELASTICITY OF SUBSTITUTION WITH MULTIPLICATIVE ERROR USING   BAYESIAN APPROACH
################################################################################
gamma = 1; delta=0.6; rho=0.5;nu=1.1
set.seed(429)
x1 = runif(50,0,1); x2 = runif(50,0,1)
set.seed(431)
u = rnorm(50,0,1)
y  = gamma*(delta*x1^(-rho)+(1-delta)*x2^(-rho))^(-nu/rho)*exp(u)
B_ces = data.frame(y,x1 ,x2)
write.csv(B_ces, "B_ces.csv")
x1_star=log(x1); x2_star=log(x2);x3_star=log(x1)^2
x4_star=log(x2)^2; x5_star=log(x1)*log(x2)
x = cbind(1,x1_star,x2_star,x3_star,x4_star,x5_star)
x = as.matrix(x)
```

```
y_star = log(y)

y_star = as.matrix(y_star)

BayesCES=data.frame(y_star,x1_star, x2_star, x3_star, x4_star,x5_star)

head(BayesCES)

write.csv(BayesCES, "BayesCES.csv")

lambda = solve(t(x)%*%x)%*%(t(x)%*%ystar)

lambda

#<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<

N = length(x1); k = 6

v = N-k

ssqrOLS = (t(ystar-x%*%lambda)%*%(y_star-x%*%lambda))/v

ssqrOLS

h=1/5000^2

h^-1

########################################################################

##    THE PRIOR   #########

#<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<

## FOR lambda ~N(lambdaprior, V_prior)

# The prior mean

lambda_prior = rbind(0.0, 0.27, 0.33, 5, 1.2, 2.1)

lambda_prior = as.matrix(lambda_prior)
```

```
lambda

 the var-cov matrix

V_prior =
matrix(c(5^2,0,0,0,0,0,0,25^2,0,0,0,0,0,0,5000^2,0,0,0,0,0,0,1000^2,0,0,0,0,0,0,10000^2,0
,0,0,0,0,0,20^2), ncol=6, nrow=6, byrow=TRUE)

V_prior

###   FOR h~G(s_prior^2, vprior)

ssqr_prior = 1/5000^2

vprior = 5

##############################################################

#   THE POSTERIOR DENSITY

#################################################################

V_post = solve%(solve%(Vprior)+%%(drop(h)*t(x)%*%(x)))

V_post

post = V_post%*%((solve(V_prior)%*%lambda_prior)+(h*t(x)%*%y_star))

post

############################################################################

# FOR h

vpost = N+vprior

ssqr_post = ((t(y_star-(x%*%lambda))%*%(y_star-(x%*%lambda))) +
vprior*ssqr_prior)/vpost

ssqr_post
```

ssqr_post_inv = ssqr_post^-1

ssqr_post_inv

###

METROPOLIS-HASTINGS ALGORITHM

##

Now our target is the posterior density of B and h.We have to use the log of the posterior instead of the actual posterior for computational reasons.

##

This is the actual Metropolis-–within Gibbs algorithm

##

create empty containers

itr = 5000

sampled.alpha0 = numeric(itr)

sampled.alpha1 = numeric(itr)

sampled.alpha2 = numeric(itr)

sampled.beta11 = numeric(itr)

sampled.beta22 = numeric(itr)

sampled.beta12 = numeric(itr)

sampled.h = numeric(itr)

##

Initials values for the Gibbs sampler

```
current.lambda = rbind(5,0.5,6.7,0.9,2.3,40)

current.lambda = as.matrix(current.lambda)

current.lambda

current.h = 1

sampled.alpha0[1] = current.lambda[1,]

sampled.alpha1[1] = current.lambda[2,]

sampled.alpha2[1] = current.lambda[3,]

sampled.beta11[1] = current.lambda[4,]

sampled.beta22[1] = current.lambda[5,]

sampled.beta12[1] = current.lambda[6,]

sampled.h[1] = current.h

########################################################################

 begin the loop  to execute the Metropolis Algorithm

#########################################################################

for (i in 2:itr){

  # draw for lambda/y, h from a multivariate denisty

  library(MASS)

  current.h = 1

  V_post = solve(solve(V%%prior)+*t(x)%*%(x)))

  V_post

  lambda_post = V_post%*%((solve(V_prior)%*%+(current.h*t(x)%*%y_star))
```

```
lambda_post

current.lambda = mvrnorm(1, post, V_post)

current.lambda = as.matrix(current.lambda)

## draws for h/y,lambda from a Gamma denisty

vpost = N+vprior

ssqr_post = ((t(y_star-(x%*%clambda))%*%(y_star-(x%*%current.lambda))) + vprior*ssqr%%prior)/vpost

ssqr_post

ssqr_post_inv = qr_post^-1

current.h = rgamma(1, ssqrpostinv, vpost)

##############################################################################

set storgae containers

##############################################################################

sampled.alpha0[i] = current.lambda[1,]

sampled.alpha1[i] = current.lambda[2,]

sampled.alpha2[i] = current.lambda[3,]

sampled.beta11[i] = current.lambda[4,]

sampled.beta22[i] = current.lambda[5,]

sampled.beta12[i] = current.lambda[6,]

sampled.h[i] =current.h

}
```

```
# Trace plots #before burnin
plot(sampled.alpha0, type = 'l')
mean(sampled.alpha0)
qqnorm(sampled.alpha0)
plot(sampled.alpha1, type = 'l')
mean(sampled.alpha1)
qqnorm(sampled.alpha1)
plot(sampled.alpha2, type = 'l')
mean(sampled.alpha2)
qqnorm(sampled.alpha2)
plot(sampled.beta11, type = 'l')
qqnorm(sampled.beta11)
mean(sampled.beta11)
plot(sampled.beta22, type = 'l')
mean(sampled.beta22)
qqnorm(sampled.beta22)
plot(sampled.h, type = 'l')
mean(sampled.h)
qqnorm(sampled.h)
##    removing the effects of the initial values
burnin=1000
```

```
final.alpha0 = sampled.alpha0[-(1:burnin)]

final.alpha1 = sampled.alpha1[-(1:burnin)]

final.alpha2 = sampled.alpha2[-(1:burnin)]

final.beta11 = sampled.beta11[-(1:burnin)]

final.beta22 = sampled.beta22[-(1:burnin)]

final.beta12 = sampled.beta12[-(1:burnin)]

final.h = sampled.h[-(1:burnin)]

######################################################

#   THE ACTUAL ESTIMATES OF THE CES MODEL

######################################################

final.gamma = exp(mean(final.alpha0))

final.gamma

final.delta = mean(final.alpha1)/(mean(final.alpha1)+mean(final.alpha2))

final.delta

final.rho =
(mean(final.beta12)*(mean(final.alpha1)+mean(final.alpha2)))/(mean(final.alpha1)*mean(
final.alpha2))

final.rho

final.nu = mean(final.alpha1)+mean(final.alpha2)

final.nu

####################################################################
```

```
#trace plot #after burnin

plot(final.alpha0, type = 'l')

mean(final.alpha0)

qqnorm(final.alpha0)

plot(final.alpha1, type = 'l')

mean(final.alpha1)

qqnorm(final.alpha1)

plot(final.alpha2, type = 'l')

mean(final.alpha2)

qqnorm(final.alpha2)

plot(final.beta11, type = 'l')

mean(final.beta11)

qqnorm(final.beta11)

plot(final.beta22, type = 'l')

mean(final.beta22)

qqnorm(final.beta22)

plot(final.beta12, type = 'l')

mean(final.beta12)

qqnorm(final.beta12)

plot(final.h, type = 'l')

mean(final.h)
```

THE HISTOGRAM PLOTS

```
hist(final.alpha0, xlab="x", ylab="y", col = "red", prob = T, main = "HISTORGAM OF alpha0")

lines(density(final.alpha0, col="black"))

plot(density(final.alpha0, xlab="x", ylab="y", col="black"))

hist(final.alpha1, xlab="x", ylab="y", col = "purple",main = "HISTORGAM OF alpha1")

hist(final.alpha2, xlab="x", ylab="y", col = "blue",main = "HISTORGAM OF alpha2")

hist(final.beta11, xlab="x", ylab="y", col = "cyan",main = "HISTORGAM OF beta11")

hist(final.beta22, xlab="x", ylab="y", col = "darkblue",main = "HISTORGAM OF beta22")

hist(final.beta12, xlab="x", ylab="y", col = "pink",main = "HISTORGAM OF beta12")

hist(final.h)

### SCATTER PLOT OF THE ESTIMATES

plot(final.alpha0)

plot(final.alpha1, col = "blue")

plot(final.alpha2)

plot(final.beta11)

plot(final.beta22)

plot(final.beta12, col="red")

############################################################

## USING CODA
```

```
library('coda')

alpha0.itr=mcmc(final.alpha0)

summary(alpha0.itr)

plot(alpha0.itr, col="pink")

title('alpha0',xlab='itr', ylab='alpha0.itr')


alpha1.itr=mcmc(final.alpha1)

summary(alpha1.itr)

plot(alpha1.itr, col="cyan")

title('alpha1',xlab='itr', ylab='alpha1.itr')


alpha2.itr=mcmc(final.alpha2)

summary(alpha2.itr)

plot(alpha2.itr, col="purple")

title('alpha2',xlab='itr', ylab='alpha2.itr')


beta11.itr=mcmc(final.beta11)

summary(beta11.itr)

plot(beta11.itr, col="purple")

title('beta11',xlab='itr', ylab='beta11.itr')


beta22.itr=mcmc(final.beta22)
```

```
summary(beta22.itr)

plot(beta22.itr, col="purple")

title('beta22',xlab='itr', ylab='beta22.itr')


beta12.itr=mcmc(final.beta12)

summary(beta12.itr)

plot(beta12.itr, col="purple")

title('beta12',xlab='itr', ylab='beta12.itr')


h.itr=mcmc(final.h)

summary(h.itr)

plot(h.itr, col="red")

title('h',xlab='itr', ylab='h.itr')

# p = data.frame(x,y_star)

# curve(dnorm(p, mean=mean(final.alpha0), sd=sd(final.aplha0)))
```

APPENDIX B

Data simulated for Cobb Douglas production function with Multiplicative error

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} e^u$) For 50 samples size.

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	1.3050012	6.4221807	6.5905422
2	3.3806409	0.9486795	5.6155654
3	3.2728793	1.5468497	2.1696898
4	1.4250526	2.524154	5.3327284
5	2.9518838	1.2684679	3.1110081
6	1.2387654	5.7422982	5.0624904
7	0.7185569	0.8581459	1.4940462
8	5.9390252	2.5244748	4.1882373
9	0.1166438	1.8583013	1.2209489
10	0.1447515	3.5699373	3.2681288
11	15.704889	21.670454	29.70356
12	23.709831	16.289003	21.433996
13	5.0749935	1.6356634	2.4815247
14	17.299354	24.467854	24.883666

15	9.7178787	11.742482	13.231317
16	7.5001724	6.8496278	8.3615224
17	2.6620037	0.3480224	0.7504386
18	6.1980373	2.9847535	4.7371306
19	20.298968	11.435497	19.589084
20	3.4055118	17.871748	15.599602
21	1.7855777	5.2736211	5.9836257
22	1.2999163	1.008773	1.1869412
23	0.9101566	4.9613212	4.5674414
24	1.1506709	0.5761371	0.8741142
25	0.7635854	1.0779878	1.8740272
26	3.021645	6.5296055	6.7608331
27	3.6752515	5.3551207	6.069585
28	0.8184425	0.7600095	1.9288473
29	4.9309724	7.0724511	10.111883
31	7.2101898	3.8201262	5.5992625
32	0.6324711	1.1684778	1.2648464
33	10.663268	1.3101904	2.915198
34	31.109909	6.7083878	12.248276
35	11.923364	11.923308	16.338319

36	2.4809831	1.6677976	2.0132905
37	3.1723005	2.5150438	2.9284657
38	13.025498	13.979601	17.965117
39	1.5909873	1.5740767	2.4431929
40	3.9235649	5.877161	7.349952
41	2.2958819	2.3717087	4.3309551
42	22.975888	10.538059	17.76221
43	3.2415918	1.3649049	2.4400398
44	5.7792871	0.1254126	0.4152731
45	8.1713097	23.876815	22.061865
46	4.1775316	1.309937	2.2560055
47	7.793092	0.4429907	1.1607517
48	0.4251882	2.4614155	4.3742058
49	1.4009565	0.8663625	1.3104511
50	22.627077	22.827593	24.075672

Data simulated for Cobb Douglas production function with Multiplicative error

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} e^u$) when N=50

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	1.3050012	6.4221807	6.5905422
2	3.3806409	0.9486795	5.6155654
3	3.2728793	1.5468497	2.1696898
4	1.4250526	2.524154	5.3327284
5	2.9518838	1.2684679	3.1110081
6	1.2387654	5.7422982	5.0624904
7	0.7185569	0.8581459	1.4940462
8	5.9390252	2.5244748	4.1882373
9	0.1166438	1.8583013	1.2209489
10	0.1447515	3.5699373	3.2681288
11	15.704889	21.670454	29.70356
12	23.709831	16.289003	21.433996
13	5.0749935	1.6356634	2.4815247
14	17.299354	24.467854	24.883666
15	9.7178787	11.742482	13.231317

16	7.5001724	6.8496278	8.3615224
17	2.6620037	0.3480224	0.7504386
18	6.1980373	2.9847535	4.7371306
19	20.298968	11.435497	19.589084
20	3.4055118	17.871748	15.599602
21	1.7855777	5.2736211	5.9836257
22	1.2999163	1.008773	1.1869412
23	0.9101566	4.9613212	4.5674414
24	1.1506709	0.5761371	0.8741142
25	0.7635854	1.0779878	1.8740272
26	3.021645	6.5296055	6.7608331
27	3.6752515	5.3551207	6.069585
28	0.8184425	0.7600095	1.9288473
29	4.9309724	7.0724511	10.111883
31	7.2101898	3.8201262	5.5992625
32	0.6324711	1.1684778	1.2648464
33	10.663268	1.3101904	2.915198
34	31.109909	6.7083878	12.248276
35	11.923364	11.923308	16.338319
36	2.4809831	1.6677976	2.0132905

37	3.1723005	2.5150438	2.9284657
38	13.025498	13.979601	17.965117
39	1.5909873	1.5740767	2.4431929
40	2.2958819	2.3717087	4.3309551
41	22.975888	10.538059	17.76221
42	3.2415918	1.3649049	2.4400398
43	5.7792871	0.1254126	0.4152731
44	8.1713097	23.876815	22.061865
45	4.1775316	1.309937	2.2560055
46	1.9284974	2.7381445	3.12617
47	7.793092	0.4429907	1.1607517
48	0.4251882	2.4614155	4.3742058
49	1.4009565	0.8663625	1.3104511
50	22.627077	22.827593	24.075672
51	4.1118186	3.4423125	4.4527828
52	4.1848483	9.1291326	8.8302605
53	24.845877	14.048006	20.787239
54	40.772592	51.391524	65.983688
55	11.979898	4.1599108	7.772756
56	28.435173	17.227021	23.093413

57	3.2448588	2.5315681	3.3629871
58	15.467622	12.375996	14.344881
59	4.1049173	2.396232	4.0538514
60	15.75343	9.7913189	12.506897
61	2.5220597	0.1311515	0.4438284
62	4.4861215	1.2766629	2.0504212
63	4.9484562	2.331525	3.985216
64	11.790106	8.2292567	10.586462
65	14.756626	34.933946	32.22165
66	8.2432681	28.180636	32.184311
67	3.8844428	10.71204	16.274519
68	6.5546795	3.91162	5.2669665
69	8.5212156	2.0330729	4.4988011
70	3.1933737	0.9997919	1.5443547
71	7.7467172	1.992097	3.4430818
72	2.3727342	1.7208927	2.2704766
73	3.0316195	2.9721437	3.6558229
74	4.2090489	4.9207746	6.875331
75	1.2571809	0.181264	0.3468572
76	42.372686	24.805797	32.462332

77	2.2913268	1.0817459	1.7349518
78	6.119058	7.5521036	9.0149899
79	0.9362999	1.2619797	1.6437179
80	46.082287	12.362638	24.029043
81	10.310606	13.077201	22.465039
82	9.6746895	5.2328161	7.1511316
83	4.6830787	6.9204495	8.4305871
84	7.4239819	15.039933	18.270569
85	17.052048	5.3379436	7.8797732
86	4.9449809	0.4305529	1.1191055
87	1.932585	17.014165	16.229872
88	115.92614	39.565472	58.45604
89	1.950488	0.8779856	1.2381824
90	1.5975815	2.1006575	3.127278
91	6.0779807	4.2444754	5.8517012
92	2.4310325	8.6402582	7.5007562
93	1.8330834	4.2942563	8.7218083
94	18.152933	22.682997	26.687491
95	16.471633	10.957555	13.382371
96	1.6103198	2.3160871	2.9025375

97	2.308563	0.8160045	1.1712289
98	12.343075	15.15644	21.173305
99	5.9533055	11.981183	12.046095
100	16.482381	1.6152371	4.4632705

Data simulated for Cobb Douglas production function with Multiplicative error

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} e^{u}$) when N=150

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	1.3050012	6.4221807	6.5905422
2	3.3806409	0.9486795	5.6155654
3	3.2728793	1.5468497	2.1696898
4	1.4250526	2.524154	5.3327284
5	2.9518838	1.2684679	3.1110081
6	1.2387654	5.7422982	5.0624904
7	0.7185569	0.8581459	1.4940462
8	5.9390252	2.5244748	4.1882373
9	0.1166438	1.8583013	1.2209489
10	0.1447515	3.5699373	3.2681288
11	15.704889	21.670454	29.70356
12	23.709831	16.289003	21.433996
13	5.0749935	1.6356634	2.4815247
14	17.299354	24.467854	24.883666

15	9.7178787	11.742482	13.231317
16	7.5001724	6.8496278	8.3615224
17	2.6620037	0.3480224	0.7504386
18	6.1980373	2.9847535	4.7371306
19	20.298968	11.435497	19.589084
20	3.4055118	17.871748	15.599602
21	1.7855777	5.2736211	5.9836257
22	1.2999163	1.008773	1.1869412
23	0.9101566	4.9613212	4.5674414
24	1.1506709	0.5761371	0.8741142
25	0.7635854	1.0779878	1.8740272
26	3.021645	6.5296055	6.7608331
27	3.6752515	5.3551207	6.069585
28	0.8184425	0.7600095	1.9288473
29	4.9309724	7.0724511	10.111883
31	7.2101898	3.8201262	5.5992625
32	0.6324711	1.1684778	1.2648464
33	10.663268	1.3101904	2.915198
34	31.109909	6.7083878	12.248276
35	11.923364	11.923308	16.338319

36	2.4809831	1.6677976	2.0132905
37	3.1723005	2.5150438	2.9284657
38	13.025498	13.979601	17.965117
39	1.5909873	1.5740767	2.4431929
40	3.9235649	5.877161	7.349952
41	2.2958819	2.3717087	4.3309551
42	22.975888	10.538059	17.76221
43	3.2415918	1.3649049	2.4400398
44	5.7792871	0.1254126	0.4152731
45	8.1713097	23.876815	22.061865
46	4.1775316	1.309937	2.2560055
47	1.9284974	2.7381445	3.12617
48	7.793092	0.4429907	1.1607517
49	0.4251882	2.4614155	4.3742058
50	1.4009565	0.8663625	1.3104511
51	22.627077	22.827593	24.075672
52	4.1118186	3.4423125	4.4527828
53	4.1848483	9.1291326	8.8302605
54	24.845877	14.048006	20.787239
55	40.772592	51.391524	65.983688

56	11.979898	4.1599108	7.772756
57	28.435173	17.227021	23.093413
58	3.2448588	2.5315681	3.3629871
59	15.467622	12.375996	14.344881
60	4.1049173	2.396232	4.0538514
61	15.75343	9.7913189	12.506897
62	2.5220597	0.1311515	0.4438284
63	4.4861215	1.2766629	2.0504212
64	4.9484562	2.331525	3.985216
65	11.790106	8.2292567	10.586462
66	14.756626	34.933946	32.22165
67	8.2432681	28.180636	32.184311
68	3.8844428	10.71204	16.274519
69	6.5546795	3.91162	5.2669665
70	8.5212156	2.0330729	4.4988011
71	3.1933737	0.9997919	1.5443547
72	7.7467172	1.992097	3.4430818
73	2.3727342	1.7208927	2.2704766
74	3.0316195	2.9721437	3.6558229
75	4.2090489	4.9207746	6.875331

76	1.2571809	0.181264	0.3468572
77	42.372686	24.805797	32.462332
78	2.2913268	1.0817459	1.7349518
79	6.119058	7.5521036	9.0149899
80	0.9362999	1.2619797	1.6437179
81	46.082287	12.362638	24.029043
82	10.310606	13.077201	22.465039
83	9.6746895	5.2328161	7.1511316
84	4.6830787	6.9204495	8.4305871
85	7.4239819	15.039933	18.270569
86	17.052048	5.3379436	7.8797732
87	4.9449809	0.4305529	1.1191055
88	1.932585	17.014165	16.229872
89	115.92614	39.565472	58.45604
90	1.950488	0.8779856	1.2381824
91	1.5975815	2.1006575	3.127278
92	6.0779807	4.2444754	5.8517012
93	2.4310325	8.6402582	7.5007562
94	1.8330834	4.2942563	8.7218083
95	18.152933	22.682997	26.687491

96	16.471633	10.957555	13.382371
97	1.6103198	2.3160871	2.9025375
98	2.308563	0.8160045	1.1712289
99	12.343075	15.15644	21.173305
100	5.9533055	11.981183	12.046095
101	16.482381	1.6152371	4.4632705
102	3.6377071	2.5558327	3.1396226
103	18.575017	1.7108874	3.6216003
105	2.3748916	1.8076188	2.4644719
106	23.518242	11.393107	16.671919
107	4.5066107	5.3590117	7.084533
108	16.463104	18.149822	23.262149
109	26.978732	26.587976	28.648152
110	0.6814986	1.3827036	1.3827892
111	7.5261748	26.897028	24.788304
112	19.143616	14.564764	23.960931
113	3.1650544	3.1253114	3.4226159
114	11.287375	24.561883	34.467964
115	6.1907238	1.7377248	2.9722295
116	1.285089	5.9253018	4.9783908

117	4.7864717	25.403857	21.097774
118	8.6016823	11.758634	14.099025
119	19.321395	12.410895	15.384224
120	19.390709	21.768106	25.393142
121	31.971895	29.845719	35.735434
122	29.42084	21.277116	30.446929
123	5.0922798	0.9040281	2.047881
124	13.737984	5.3418213	8.84091
125	12.511477	5.261552	6.8063919
126	3.3085413	1.9070323	3.1047053
127	20.636129	5.6888503	9.6129347
128	2.8381669	0.9076053	2.1074394
129	6.9666752	6.1238637	13.164182
130	3.5352101	1.9660885	2.4701597
131	8.1643196	8.458108	9.1139783
132	2.5927425	3.9096513	3.9893795
133	51.531286	55.764837	56.08541
134	7.2099145	1.6658564	3.0684061
135	1.5816747	1.1817277	1.71033
136	1.5550496	1.7777615	2.2456363

137	0.2078518	2.2642422	2.3998578
138	4.5541058	1.5557539	2.4257884
139	17.354549	13.992525	19.230546
140	20.160694	25.320333	26.328202
141	25.0906	3.6956266	6.6745865
142	25.029612	22.736744	24.450965
143	0.9506363	1.7578895	1.7688096
144	3.5516633	1.0856802	1.7071614
145	1.2085901	0.1772278	0.3661742
146	46.153968	56.374295	58.878729
147	6.8389795	2.8744876	5.3155921
148	24.991068	32.154635	35.363308
149	11.444049	10.365895	12.645178
150	12.480869	8.5661406	11.825258

Data simulated for Cobb Douglas production function with Multiplicative error

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} e^u$) when N=250

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	1.3050012	6.4221807	6.5905422
2	3.3806409	0.9486795	5.6155654
3	3.2728793	1.5468497	2.1696898
4	1.4250526	2.524154	5.3327284
5	2.9518838	1.2684679	3.1110081
6	1.2387654	5.7422982	5.0624904
7	0.7185569	0.8581459	1.4940462
8	5.9390252	2.5244748	4.1882373
9	0.1166438	1.8583013	1.2209489
10	0.1447515	3.5699373	3.2681288
11	15.704889	21.670454	29.70356
12	23.709831	16.289003	21.433996
13	5.0749935	1.6356634	2.4815247
14	17.299354	24.467854	24.883666
15	9.7178787	11.742482	13.231317

16	7.5001724	6.8496278	8.3615224
17	2.6620037	0.3480224	0.7504386
18	6.1980373	2.9847535	4.7371306
19	20.298968	11.435497	19.589084
20	3.4055118	17.871748	15.599602
21	1.7855777	5.2736211	5.9836257
22	1.2999163	1.008773	1.1869412
23	0.9101566	4.9613212	4.5674414
24	1.1506709	0.5761371	0.8741142
25	0.7635854	1.0779878	1.8740272
26	3.021645	6.5296055	6.7608331
27	3.6752515	5.3551207	6.069585
28	0.8184425	0.7600095	1.9288473
29	4.9309724	7.0724511	10.111883
31	7.2101898	3.8201262	5.5992625
32	0.6324711	1.1684778	1.2648464
33	10.663268	1.3101904	2.915198
34	31.109909	6.7083878	12.248276
35	11.923364	11.923308	16.338319
36	2.4809831	1.6677976	2.0132905

37	3.1723005	2.5150438	2.9284657
38	13.025498	13.979601	17.965117
39	1.5909873	1.5740767	2.4431929
40	3.9235649	5.877161	7.349952
41	2.2958819	2.3717087	4.3309551
42	22.975888	10.538059	17.76221
43	3.2415918	1.3649049	2.4400398
44	5.7792871	0.1254126	0.4152731
45	8.1713097	23.876815	22.061865
46	4.1775316	1.309937	2.2560055
47	1.9284974	2.7381445	3.12617
48	7.793092	0.4429907	1.1607517
49	0.4251882	2.4614155	4.3742058
50	1.4009565	0.8663625	1.3104511
51	22.627077	22.827593	24.075672
52	4.1118186	3.4423125	4.4527828
53	4.1848483	9.1291326	8.8302605
54	24.845877	14.048006	20.787239
55	40.772592	51.391524	65.983688
56	11.979898	4.1599108	7.772756

57	28.435173	17.227021	23.093413
58	3.2448588	2.5315681	3.3629871
59	15.467622	12.375996	14.344881
60	4.1049173	2.396232	4.0538514
61	15.75343	9.7913189	12.506897
62	2.5220597	0.1311515	0.4438284
63	4.4861215	1.2766629	2.0504212
64	4.9484562	2.331525	3.985216
65	11.790106	8.2292567	10.586462
66	14.756626	34.933946	32.22165
67	8.2432681	28.180636	32.184311
68	3.8844428	10.71204	16.274519
69	6.5546795	3.91162	5.2669665
70	8.5212156	2.0330729	4.4988011
71	3.1933737	0.9997919	1.5443547
72	7.7467172	1.992097	3.4430818
73	2.3727342	1.7208927	2.2704766
74	3.0316195	2.9721437	3.6558229
75	4.2090489	4.9207746	6.875331
76	1.2571809	0.181264	0.3468572

77	42.372686	24.805797	32.462332
78	2.2913268	1.0817459	1.7349518
79	6.119058	7.5521036	9.0149899
80	0.9362999	1.2619797	1.6437179
81	46.082287	12.362638	24.029043
82	10.310606	13.077201	22.465039
83	9.6746895	5.2328161	7.1511316
84	4.6830787	6.9204495	8.4305871
85	7.4239819	15.039933	18.270569
86	17.052048	5.3379436	7.8797732
87	4.9449809	0.4305529	1.1191055
88	1.932585	17.014165	16.229872
89	115.92614	39.565472	58.45604
90	1.950488	0.8779856	1.2381824
91	1.5975815	2.1006575	3.127278
92	6.0779807	4.2444754	5.8517012
93	2.4310325	8.6402582	7.5007562
94	1.8330834	4.2942563	8.7218083
95	18.152933	22.682997	26.687491
96	16.471633	10.957555	13.382371

97	1.6103198	2.3160871	2.9025375
98	2.308563	0.8160045	1.1712289
99	12.343075	15.15644	21.173305
100	5.9533055	11.981183	12.046095
101	16.482381	1.6152371	4.4632705
102	3.6377071	2.5558327	3.1396226
103	18.575017	1.7108874	3.6216003
105	2.3748916	1.8076188	2.4644719
106	23.518242	11.393107	16.671919
107	4.5066107	5.3590117	7.084533
108	16.463104	18.149822	23.262149
109	26.978732	26.587976	28.648152
110	0.6814986	1.3827036	1.3827892
111	7.5261748	26.897028	24.788304
112	19.143616	14.564764	23.960931
113	3.1650544	3.1253114	3.4226159
114	11.287375	24.561883	34.467964
115	6.1907238	1.7377248	2.9722295
116	1.285089	5.9253018	4.9783908
117	4.7864717	25.403857	21.097774

118	8.6016823	11.758634	14.099025
119	19.321395	12.410895	15.384224
120	19.390709	21.768106	25.393142
121	31.971895	29.845719	35.735434
122	29.42084	21.277116	30.446929
123	5.0922798	0.9040281	2.047881
124	13.737984	5.3418213	8.84091
125	12.511477	5.261552	6.8063919
126	3.3085413	1.9070323	3.1047053
127	20.636129	5.6888503	9.6129347
128	2.8381669	0.9076053	2.1074394
129	6.9666752	6.1238637	13.164182
130	3.5352101	1.9660885	2.4701597
131	8.1643196	8.458108	9.1139783
132	2.5927425	3.9096513	3.9893795
133	51.531286	55.764837	56.08541
134	7.2099145	1.6658564	3.0684061
135	1.5816747	1.1817277	1.71033
136	1.5550496	1.7777615	2.2456363
137	0.2078518	2.2642422	2.3998578

138	4.5541058	1.5557539	2.4257884
139	17.354549	13.992525	19.230546
140	20.160694	25.320333	26.328202
141	25.0906	3.6956266	6.6745865
142	25.029612	22.736744	24.450965
143	0.9506363	1.7578895	1.7688096
144	3.5516633	1.0856802	1.7071614
145	1.2085901	0.1772278	0.3661742
146	46.153968	56.374295	58.878729
147	6.8389795	2.8744876	5.3155921
148	24.991068	32.154635	35.363308
149	11.444049	10.365895	12.645178
150	12.480869	8.5661406	11.825258
151	28.862052	4.9521029	8.8708489
152	6.6286804	11.781907	13.487623
153	2.9909887	8.0546734	7.6387179
154	9.1865725	2.5674512	3.92469
155	5.5339382	3.2607008	5.9713222
156	0.735294	1.1337369	1.3083387
157	2.4381353	2.168984	2.6386268

158	1.0172438	0.2881161	0.4579736
159	15.158867	3.3166083	7.4951497
160	3.6715713	11.109621	14.127761
161	2.0680366	3.9144445	3.8140644
162	0.873379	0.3548709	0.5707778
163	14.339214	18.550709	21.057239
164	4.2054148	3.746279	5.1232596
165	3.3611281	10.344383	10.065879
166	2.5934991	2.5638772	3.6108833
167	3.8057039	5.8383117	7.116014
168	4.1969934	2.1663782	2.8675255
169	27.138763	21.829705	32.200869
170	2.0867629	1.8332457	3.3909831
171	9.8241633	12.542938	12.488682
172	1.9177037	1.5866731	2.3741644
173	3.327809	2.2127909	3.2113895
174	3.0616443	7.466383	6.782499
175	5.957434	8.3406876	12.607088
176	32.977726	30.195335	33.243832
177	7.1091927	7.7847776	8.2257436

178	9.3193009	24.045763	23.832961
179	5.2161288	1.9926294	3.914889
180	5.0183156	2.9029441	3.7999734
181	5.7107869	11.54011	18.356377
182	16.783739	8.2725751	12.539746
183	15.561841	8.4528869	10.63583
184	7.2542287	2.8470453	6.2313703
185	0.6188387	1.2200781	1.6777975
186	3.6487917	5.2213013	6.2538896
187	22.761927	15.087113	17.836563
188	55.572598	60.776009	79.190752
189	5.5451729	7.6788854	9.27613
190	5.2064253	10.892785	12.615507
191	24.686098	20.82835	23.688936
192	6.5003202	3.2991593	4.4558057
193	6.1382873	1.3776182	2.7482195
194	26.983088	2.7731512	7.1348845
195	5.3527451	0.5388447	1.0739846
196	0.3187782	2.3082335	1.7304967
197	0.7630661	1.899624	2.4443978

198	2.8765522	5.0040337	5.1887638
199	1.2522444	1.5909926	1.6336996
200	3.5749258	1.1067502	1.6983275
201	0.8497229	3.4621875	3.4989117
202	1.4334117	3.5008129	3.2483286
203	41.833754	10.498973	18.998034
204	4.7592757	9.3533686	11.072586
205	32.000558	37.501569	41.795304
206	24.035625	22.458986	31.332662
207	4.7014055	0.8611415	1.469722
208	16.445743	16.868224	17.72187
209	13.348187	28.700215	29.914732
210	0.2559678	2.9608604	2.6777247
211	4.0638246	1.6873976	4.7364007
212	1.2840921	3.4154206	3.1217314
213	13.397493	11.946935	13.383863
214	0.3599687	36.430031	16.965134
215	4.2587948	6.941884	7.0264344
216	1.6061611	0.9850854	1.2874317
217	11.20333	0.9701038	2.8036161

218	2.8601924	3.0550974	3.0984637
219	4.9803348	25.684294	19.925814
220	3.8880737	4.9258633	5.6930445
221	8.8851106	8.2403081	8.6861917
222	1.0306182	0.780371	1.0244486
223	6.0756629	1.3023057	2.1995407
224	2.2849159	7.9283582	6.694315
225	8.2134742	20.183961	17.622633
226	6.3736047	8.1792616	8.5238798
227	4.7992917	4.712616	5.6561872
228	6.4284759	7.8747496	9.139608
229	6.7263566	2.7697557	8.4231412
230	47.644753	53.528486	62.463459
231	3.827245	18.933778	16.803959
232	29.322829	18.39735	28.294918
233	1.1979835	5.0461058	5.2199575
234	0.7534151	1.4877593	1.95611
235	4.7144116	8.1386786	9.2285692
236	5.6972868	3.4498389	4.833998
237	10.604857	15.725605	16.46021

238	11.44227	9.2963265	12.101785
238	3.7712654	5.0743422	5.5803964
240	0.8636279	0.8873888	0.9731304
241	3.994707	3.6286332	4.2208607
242	2.4213611	2.0021435	3.1089334
243	36.472883	26.376727	31.071331
244	3.4662948	16.126445	14.722236
245	22.869048	16.641492	23.910821
246	8.9625393	1.599611	2.8486574
247	17.226524	12.75982	14.697867
248	9.6287464	8.8288301	10.163102
249	1.7727517	2.2408464	2.7567537
250	9.911306	9.2014812	9.7072588

Data simulated for Cobb Douglas production function with Multiplicative error term

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} e^u$) when N=500

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	1.3050012	6.4221807	6.5905422
2	3.3806409	0.9486795	5.6155654
3	3.2728793	1.5468497	2.1696898
4	1.4250526	2.524154	5.3327284
5	2.9518838	1.2684679	3.1110081
6	1.2387654	5.7422982	5.0624904
7	0.7185569	0.8581459	1.4940462
8	5.9390252	2.5244748	4.1882373
9	0.1166438	1.8583013	1.2209489
10	0.1447515	3.5699373	3.2681288
11	15.704889	21.670454	29.70356
12	23.709831	16.289003	21.433996
13	5.0749935	1.6356634	2.4815247
14	17.299354	24.467854	24.883666
15	9.7178787	11.742482	13.231317

16	7.5001724	6.8496279	8.3615224
17	2.6620037	0.3480224	0.7504386
18	6.1980373	2.9847535	4.7371306
19	20.298968	11.435497	19.589084
20	3.4055118	17.871748	15.599602
21	1.7855777	5.2736211	5.9836257
22	1.2999163	1.008773	1.1869412
23	0.9101566	4.9613212	4.5674414
24	1.1506709	0.5761371	0.8741142
25	0.7635854	1.0779878	1.8740273
26	3.021645	6.5296055	6.7608331
27	3.6752515	5.3551207	6.069585
28	0.8184425	0.7600095	1.9288473
29	4.9309724	7.0724511	10.111883
31	7.2101898	3.8201262	5.5992625
32	0.6324711	1.1684778	1.2648464
33	10.663267	1.3101904	2.915198
34	31.109909	6.7083878	12.248276
35	11.923363	11.923308	16.338319
36	2.4809831	1.6677976	2.0132905

37	3.1723005	2.5150438	2.9284657
38	13.025498	13.979601	17.965117
39	1.5909873	1.5740767	2.4431929
40	3.9235649	5.877161	7.349952
41	2.2958819	2.3717087	4.3309551
42	22.975888	10.538059	17.76221
43	3.2415918	1.3649049	2.4400398
44	5.7792871	0.1254126	0.4152731
45	8.1713097	23.876815	22.061865
46	4.1775316	1.309937	2.2560055
47	1.9284974	2.7381446	3.12617
48	7.793092	0.4429907	1.1607517
49	0.4251882	2.4614155	4.3742058
50	1.4009565	0.8663625	1.3104511
51	22.627077	22.827593	24.075672
52	4.1118186	3.4423125	4.4527828
53	4.1848483	9.1291326	8.8302605
54	24.845877	14.048006	20.787239
55	40.772592	51.391524	65.983688
56	11.979898	4.1599108	7.772756

57	28.435173	17.227021	23.093413
58	3.2448588	2.5315681	3.3629871
59	15.467622	12.375996	14.34488
60	4.1049173	2.396232	4.0538514
61	15.75343	9.7913189	12.506897
62	2.5220597	0.1311515	0.4438284
63	4.4861215	1.2766629	2.0504212
64	4.9484562	2.331525	3.985216
65	11.790106	8.2292567	10.586462
66	14.756626	34.933946	32.22165
67	8.2432681	28.180636	32.184311
68	3.8844428	10.71204	16.274519
69	6.5546795	3.9116201	5.2669665
70	8.5212156	2.0330729	4.4988011
71	3.1933737	0.9997919	1.5443547
72	7.7467172	1.992097	3.4430818
73	2.3727342	1.7208927	2.2704766
74	3.0316195	2.9721437	3.6558229
75	4.2090489	4.9207746	6.875331
76	1.2571809	0.181264	0.3468572

77	42.372686	24.805797	32.462332
78	2.2913268	1.0817459	1.7349518
79	6.119058	7.5521036	9.0149899
80	0.9362999	1.2619797	1.6437179
81	46.082287	12.362637	24.029043
82	10.310606	13.0772	22.465039
83	9.6746895	5.2328161	7.1511316
84	4.6830787	6.9204495	8.4305871
85	7.4239819	15.039933	18.270569
86	17.052048	5.3379436	7.8797732
87	4.9449809	0.4305529	1.1191055
88	1.932585	17.014165	16.229872
89	115.92614	39.565472	58.45604
90	1.950488	0.8779856	1.2381824
91	1.5975815	2.1006575	3.127278
92	6.0779807	4.2444754	5.8517012
93	2.4310325	8.6402582	7.5007562
94	1.8330834	4.2942563	8.7218083
95	18.152933	22.682997	26.687491
96	16.471633	10.957555	13.382371

97	1.6103198	2.3160871	2.9025375
98	2.308563	0.8160045	1.1712289
99	12.343075	15.15644	21.173305
100	5.9533055	11.981183	12.046095
101	16.482381	1.6152371	4.4632705
102	3.6377071	2.5558327	3.1396226
103	18.575017	1.7108874	3.6216003
105	2.3748916	1.8076188	2.4644719
106	23.518242	11.393107	16.671919
107	4.5066107	5.3590117	7.084533
108	16.463104	18.149822	23.262149
109	26.978732	26.587976	28.648152
110	0.6814986	1.3827036	1.3827892
111	7.5261748	26.897028	24.788304
112	19.143616	14.564764	23.960931
113	3.1650544	3.1253114	3.4226159
114	11.287375	24.561883	34.467964
115	6.1907238	1.7377248	2.9722295
116	1.285089	5.9253018	4.9783908
117	4.7864717	25.403857	21.097774

118	8.6016823	11.758634	14.099025
119	19.321395	12.410895	15.384224
120	19.390709	21.768106	25.393142
121	31.971895	29.845719	35.735434
122	29.42084	21.277116	30.446929
123	5.0922798	0.9040281	2.047881
124	13.737984	5.3418213	8.84091
125	12.511477	5.261552	6.8063919
126	3.3085413	1.9070323	3.1047053
127	20.636129	5.6888503	9.6129348
128	2.8381669	0.9076053	2.1074394
129	6.9666752	6.1238637	13.164182
130	3.5352101	1.9660885	2.4701597
131	8.1643196	8.458108	9.1139783
132	2.5927425	3.9096513	3.9893795
133	51.531286	55.764837	56.08541
134	7.2099145	1.6658564	3.0684061
135	1.5816747	1.1817277	1.7103301
136	1.5550496	1.7777615	2.2456364
137	0.2078518	2.2642422	2.3998578

138	4.5541058	1.5557539	2.4257884
139	17.354549	13.992525	19.230546
140	20.160694	25.320333	26.328202
141	25.0906	3.6956266	6.6745865
142	25.029612	22.736744	24.450965
143	0.9506363	1.7578895	1.7688096
144	3.5516633	1.0856802	1.7071614
145	1.2085901	0.1772278	0.3661742
146	46.153968	56.374295	58.878729
147	6.8389795	2.8744876	5.3155921
148	24.991068	32.154635	35.363308
149	11.444049	10.365895	12.645178
150	12.480869	8.5661406	11.825257
151	28.862052	4.952103	8.8708489
152	6.6286804	11.781907	13.487623
153	2.9909887	8.0546734	7.6387179
154	9.1865725	2.5674512	3.92469
155	5.5339382	3.2607008	5.9713222
156	0.735294	1.1337369	1.3083387
157	2.4381353	2.168984	2.6386268

158	1.0172438	0.2881161	0.4579736
159	15.158867	3.3166083	7.4951497
160	3.6715713	11.109621	14.127761
161	2.0680366	3.9144445	3.8140644
162	0.873379	0.3548709	0.5707778
163	14.339214	18.550709	21.057239
164	4.2054148	3.746279	5.1232596
165	3.3611281	10.344383	10.065879
166	2.5934991	2.5638772	3.6108833
167	3.8057039	5.8383117	7.116014
168	4.1969934	2.1663782	2.8675255
169	27.138763	21.829705	32.200869
170	2.0867629	1.8332457	3.3909831
171	9.8241633	12.542938	12.488682
172	1.9177037	1.5866731	2.3741644
173	3.3278091	2.2127909	3.2113895
174	3.0616443	7.466383	6.782499
175	5.957434	8.3406876	12.607088
176	32.977726	30.195335	33.243832
177	7.1091927	7.7847776	8.2257436

178	9.3193009	24.045763	23.83296
179	5.2161288	1.9926294	3.914889
180	5.0183156	2.9029441	3.7999734
181	5.7107869	11.54011	18.356377
182	16.783739	8.2725751	12.539745
183	15.561841	8.4528869	10.63583
184	7.2542287	2.8470453	6.2313703
185	0.6188387	1.2200781	1.6777975
186	3.6487917	5.2213013	6.2538896
187	22.761927	15.087113	17.836563
188	55.572597	60.776009	79.190752
189	5.5451729	7.6788854	9.27613
190	5.2064253	10.892784	12.615507
191	24.686098	20.82835	23.688936
192	6.5003203	3.2991593	4.4558057
193	6.1382873	1.3776182	2.7482195
194	26.983088	2.7731512	7.1348845
195	5.3527451	0.5388447	1.0739846
196	0.3187782	2.3082335	1.7304967
197	0.7630661	1.899624	2.4443978

198	2.8765522	5.0040337	5.1887638
199	1.2522444	1.5909926	1.6336996
200	3.5749258	1.1067502	1.6983275
201	0.8497229	3.4621875	3.4989118
202	1.4334117	3.5008129	3.2483286
203	41.833754	10.498973	18.998034
204	4.7592757	9.3533687	11.072586
205	32.000558	37.501569	41.795304
206	24.035625	22.458985	31.332662
207	4.7014055	0.8611415	1.4697221
208	16.445743	16.868224	17.72187
209	13.348187	28.700215	29.914732
210	0.2559678	2.9608604	2.6777247
211	4.0638246	1.6873976	4.7364007
212	1.2840921	3.4154206	3.1217314
213	13.397493	11.946935	13.383863
214	0.3599687	36.430031	16.965134
215	4.2587948	6.941884	7.0264344
216	1.6061611	0.9850854	1.2874317
217	11.20333	0.9701038	2.8036161

218	2.8601924	3.0550974	3.0984637
219	4.9803348	25.684294	19.925814
220	3.8880737	4.9258633	5.6930445
221	8.8851106	8.2403081	8.6861917
222	1.0306182	0.780371	1.0244486
223	6.0756629	1.3023057	2.1995407
224	2.2849159	7.9283582	6.694315
225	8.2134742	20.183961	17.622633
226	6.3736047	8.1792616	8.5238798
227	4.7992917	4.712616	5.6561872
228	6.4284759	7.8747496	9.1396081
229	6.7263566	2.7697557	8.4231412
230	47.644753	53.528486	62.463459
231	3.827245	18.933778	16.803959
232	29.322829	18.39735	28.294918
233	1.1979835	5.0461058	5.2199575
234	0.7534151	1.4877593	1.95611
235	4.7144116	8.1386786	9.2285692
236	5.6972868	3.4498389	4.833998
237	10.604857	15.725605	16.46021

238	11.44227	9.2963265	12.101785
238	3.7712654	5.0743422	5.5803964
240	0.8636279	0.8873888	0.9731304
241	3.994707	3.6286332	4.2208607
242	2.4213611	2.0021435	3.1089334
243	36.472883	26.376727	31.071331
244	3.4662948	16.126445	14.722236
245	22.869048	16.641492	23.910821
246	8.9625393	1.599611	2.8486574
247	17.226524	12.75982	14.697867
248	9.6287464	8.8288301	10.163102
249	1.7727517	2.2408464	2.7567538
250	9.911306	9.2014812	9.7072588
251	1.2634283	6.6587056	6.0305402
252	19.856675	19.549719	32.067241
253	40.547462	9.2709113	14.28637
254	6.2939461	26.086527	26.301426
255	1.6167047	2.1469704	2.1830377
256	1.0440269	1.0863624	1.7272562
257	7.1312285	1.2435283	2.1499423

258	35.983031	31.711516	37.489377
259	0.4104839	2.9148041	2.3112641
260	3.2724409	1.3879207	2.8310382
261	2.0091522	0.50956	1.0860193
262	0.8169769	4.399342	3.4793716
263	16.176245	17.898913	20.676089
264	7.197486	7.3722522	8.9883128
265	0.7103852	0.7206661	1.1157815
266	3.3434411	4.197976	7.0214453
267	10.735346	1.2185177	2.7579232
268	0.6600521	9.2741183	9.6587946
269	11.221131	3.2468096	6.6878089
270	0.9204277	0.4675871	0.7862862
271	5.8138425	4.4021504	6.4423731
272	7.4926213	7.4490447	9.557745
273	8.9998522	1.9328543	3.4582761
274	1.4142623	9.0389608	7.4980477
275	5.4751136	5.4209684	6.8167252
276	11.408714	4.6787671	6.6100364
277	72.686032	55.306595	138.59357

278	46.846769	65.881807	62.57994
279	0.9223359	4.6619343	4.3119131
280	5.3107196	6.138654	7.0568831
281	19.993212	22.303627	24.453332
282	34.228292	4.9416547	10.932091
283	12.024335	0.7552084	2.7869187
284	3.7019913	15.335451	12.377438
285	1.3646478	0.6345863	0.9461147
286	22.265833	42.376673	40.932018
287	9.6171122	0.3218331	1.3972994
288	1.207346	3.8765743	3.5261732
289	13.088355	12.48285	15.209106
290	2.4924091	6.424102	6.0824532
291	2.408944	11.283612	10.392015
292	45.843283	48.088052	53.308137
293	4.0148474	14.303407	12.773916
294	18.086112	20.842203	25.504273
295	0.3089786	4.3671836	3.0676401
296	13.254331	14.18845	21.413949
297	12.41447	12.61569	13.362065

299	12.982981	19.076696	20.885606
300	9.6620727	35.920379	34.699134
301	3.431578	0.4991525	1.0970863
302	0.849821	6.0697454	4.9358344
303	37.138281	7.0443586	16.19783
304	28.260612	14.600737	31.421835
305	19.980401	0.2886627	1.1406755
306	5.916063	3.3131442	4.2153136
307	18.490897	18.061207	21.538047
308	6.7797455	13.910166	13.408725
309	17.709785	12.46515	16.5333
310	3.3396476	2.0717807	2.7625688
311	20.13996	10.723172	16.393146
312	1.9552767	0.5820983	1.1875924
313	3.3672566	3.336831	3.9106637
314	20.648514	0.3841004	1.2306418
315	19.334734	9.1935549	27.064302
316	106.01873	122.35565	123.8734
317	0.4857851	5.2225875	4.3530725
318	0.4394561	10.723307	6.2459013

319	25.009115	14.185296	23.997153
320	36.957667	7.4232435	16.005565
321	2.0727032	0.610956	1.4300895
322	125.62453	13.280697	30.705777
323	29.517149	19.172075	22.40902
324	2.8283219	28.748397	22.376844
325	15.939178	17.713085	24.474254
326	26.013349	8.8596196	15.233886
327	1.1390261	12.633572	8.2676875
328	0.5960731	6.791288	4.8624181
329	5.5583771	4.0368781	7.0253392
340	6.9122179	2.9372057	4.3698819
341	6.895181	3.5224659	4.9131015
342	3.7378313	3.5380412	6.1934899
343	4.0735393	10.471148	12.68075
344	7.8870319	5.9479035	8.2281266
345	18.413457	12.042453	19.351627
346	0.3127967	1.0312283	1.3837435
347	5.3346811	1.7441634	2.5290841
348	10.974438	1.0141238	2.3408694

349	1.6773678	7.9005978	6.4048678
350	4.0562613	7.5426835	8.082659
351	10.06727	8.1511626	13.206116
352	14.524617	1.6592396	3.5344543
353	16.789181	23.408014	23.058299
354	16.081767	7.5303269	11.128393
355	8.0430964	14.657127	14.423013
356	3.6686796	3.2669834	4.0195805
357	3.4976252	3.7752092	4.9138735
358	2.5631168	4.1362988	4.1575907
359	25.973661	11.851753	17.414768
360	3.2107151	6.4011034	6.2938034
361	17.930662	16.558369	19.394802
362	72.554018	26.944283	52.560619
363	3.6825974	3.110469	4.5871447
364	4.9286174	5.4443118	5.5645697
365	15.897365	51.212117	42.644389
366	0.6126852	1.3010516	1.6948337
367	8.922792	30.209149	26.218551
368	81.378431	52.768336	63.844307

369	2.1759782	7.7420421	9.9318506
370	46.457073	40.496094	43.916887
371	14.155586	32.914685	30.521836
372	7.8424218	20.605053	18.38443
373	40.79128	69.415964	78.340091
374	3.7237851	26.586204	22.617299
375	11.164424	7.3555458	12.097243
376	0.4390489	2.4190942	2.1317777
377	1.0523159	0.8435661	1.4970081
378	1.6759519	0.5960915	0.9448189
379	1.1696015	1.2145504	1.791548
380	11.543268	11.841426	16.832896
381	0.1932293	2.5974672	1.9425726
382	46.916278	97.536519	88.384581
383	5.7917383	8.0450275	8.8846066
384	2.8591674	12.172414	11.277331
385	2.6068782	7.173396	8.255747
386	4.5221899	3.5308967	4.9249835
387	8.86262	0.9556623	1.9441564
388	13.223547	10.431241	13.310368

389	41.065452	64.212575	69.975689
390	15.589547	12.570038	16.824251
391	4.6715882	1.7161879	3.9011617
392	2.8494571	1.7162685	2.1739344
393	15.717174	1.8507524	4.0041604
394	7.4886413	4.4379672	5.9005546
395	1.3867827	4.0338318	4.1299687
396	11.635716	21.058319	24.11458
397	4.5175592	6.7735202	6.823576
398	3.3298279	0.9611867	1.4155914
399	4.2272837	2.3060021	2.8666309
400	5.0743601	0.8460733	1.6156081
401	1.8787312	3.6818132	4.2369405
402	67.549906	8.4563	16.810271
403	5.9784954	2.3656085	4.3017101
404	5.261273	3.2871258	3.9059681
405	2.4181155	6.6637033	6.0225014
406	17.260151	1.2653246	2.6892065
407	1.0916869	3.1999061	2.9218344
408	7.9917157	1.1583699	3.1064622

409	4.9795044	8.6289419	10.856816
410	3.824313	5.5598606	6.9567667
411	37.417356	79.645833	75.012232
412	1.9755643	1.3172598	1.7561086
413	39.013474	36.105829	48.258604
414	0.0500647	11.736217	4.7913884
415	15.585962	32.516372	31.551348
416	70.984693	87.15871	88.530408
417	11.162203	14.630021	16.895469
418	4.6851827	4.2245549	6.6767754
419	0.3778088	0.6994515	0.8255764
420	0.2627459	1.6898702	1.2913359
421	0.3361221	1.0770876	1.3473599
422	10.65688	8.0336725	10.617776
423	3.0323181	1.8267428	3.2480722
424	11.385298	8.1986594	10.246472
425	8.6503864	5.4282761	11.455949
426	5.2358417	6.6936089	7.9100875
427	38.139548	29.515892	44.081016
428	13.140983	13.117204	20.377036

429	0.4380517	0.0200331	0.064274
430	1.6055961	0.8031289	1.4134081
431	6.2253097	1.7578574	2.720561
432	8.7497944	1.7008365	4.3482765
433	2.0162139	1.5768216	2.4099476
434	15.226828	13.497589	15.297939
435	35.145207	31.790368	35.423605
436	4.1785384	7.0940945	6.8936083
437	22.309572	22.832256	28.056786
438	13.04233	7.4465199	12.372679
439	10.099371	11.130993	11.324202
440	1.247567	0.2807605	0.622018
441	42.862787	44.993721	50.555994
442	39.805078	44.955916	46.553801
443	54.477332	39.135027	43.705568
445	10.259812	1.1899671	2.7060636
446	9.7088864	4.3201828	6.2197441
447	5.345733	8.832931	11.068063
448	20.407608	5.7574817	10.654198
449	10.110335	4.0387958	5.4336145

450	8.3224992	0.762654	1.9224483
451	11.771122	10.369536	16.763601
452	0.4259835	0.980439	1.2193251
453	1.2006151	0.8390575	1.2453364
454	8.7000925	9.4871617	12.913312
455	55.66016	25.645305	36.02926
456	2.7890727	6.1261945	6.0543761
457	7.1114638	16.053193	16.492005
458	3.1533268	2.9042037	3.9147565
459	10.754142	8.8043789	10.297206
460	0.1366483	3.3382863	2.2536204
461	6.6205557	8.0921124	10.504968
462	53.070189	50.103038	56.254103
463	8.4968141	5.6916134	9.0301079
464	2.380943	0.6990609	1.4449962
465	17.515905	32.85752	34.522298
466	0.3640498	2.097465	2.2212524
467	0.4459029	0.6055432	0.6904051
468	1.8488865	0.2465173	0.6914031
469	7.4096276	10.027247	10.191076

470	3.9105114	3.7791929	4.2713472
471	6.5240139	4.3195969	8.4757024
472	2.2744427	4.0358048	4.5561268
473	5.2571419	25.61628	25.17736
474	7.608393	0.2384903	0.6527463
475	3.5574579	2.991386	3.4794539
476	37.20338	20.718217	29.530411
477	4.9632424	1.5544329	2.6635068
478	12.221972	15.082789	19.228667
479	6.3878912	2.9803067	7.7783137
480	1.2363428	0.9051926	1.4374472
481	16.177137	109.90891	91.572131
482	4.2715718	4.6942605	6.8331471
483	7.8932597	14.57138	26.624588
484	2.1817622	7.4938596	7.5253401
485	1.053272	5.9795832	5.2628218
486	1.1438394	1.8767679	1.7891236
487	14.652152	8.2279507	12.304024
488	9.6136494	4.7661085	6.0246595
489	3.9267828	0.0889388	0.2609534

490	24.790961	34.448893	36.61383
491	2.9198149	1.3546976	2.671131
492	30.759857	12.351988	18.89991
493	0.3898377	0.1891527	0.279967
494	33.965818	2.1985154	6.0788384
495	5.7290032	8.9589885	8.621856
496	3.2788338	0.9784225	2.7196375
497	25.369278	42.151649	41.817141
498	17.130274	8.9569316	12.150442
499	2.694859	2.8324319	2.8839816
500	3.1668221	6.4850585	6.631099

Data simulated for Cobb Douglas production function with Additive error term

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} + u$) when N=50

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	0.8966972	4.3207465	4.499777
2	3.9726084	5.3683955	9.541888
3	8.7719229	8.0438458	8.174889
4	7.4586476	9.2511609	5.519408
5	11.136143	3.5230401	9.104479
6	2.8350893	10.565485	10.208403
7	10.305206	10.95307	2.425848
8	11.852526	8.0158034	6.77906
9	7.3343574	9.6713662	8.628992
10	8.2455079	0.8877123	8.400322
11	1.588322	5.9547049	9.989974
12	3.8516867	2.4258874	7.188984
13	1.9176462	9.846467	9.910242
14	9.5847612	5.1145804	7.660345
15	4.8229733	10.134784	14.320189

16	11.502734	6.2938691	10.636405
17	5.3397861	7.7974488	3.045382
18	10.495471	15.584288	11.840067
19	14.562114	3.9818841	10.717949
20	4.6128079	13.817292	4.772213
21	11.890113	11.845421	5.478017
22	12.299618	2.878068	1.419639
23	3.2623095	8.8654988	4.926368
24	8.7729844	0.9268447	3.462777
25	2.8449491	2.6896661	5.557128
26	5.4723625	6.5177116	12.522336
27	6.7107981	-0.143002	9.418289
28	0.2575361	5.2317551	11.147447
29	4.8407118	12.653156	8.942384
31	11.389897	5.0511846	11.820918
32	5.1678998	6.015916	9.4357
33	8.3738368	8.7099307	7.065697
34	10.001504	7.7925287	4.790747
35	6.9538724	1.1162917	5.61649
36	2.6123096	12.557268	6.976163

37	11.827448	7.9192591	7.187584
38	9.3806663	12.450812	11.245944
39	11.441905	1.0232744	3.884956
40	2.9045531	8.8365619	10.267793
41	9.8771973	4.83863	4.446689
42	6.5660554	8.8955361	2.524445
43	9.3641412	6.9624688	9.536695
44	6.7839988	12.175524	3.582742
45	10.519435	9.2725076	11.183721
46	9.4847389	9.6411865	7.959757
47	8.5385888	12.145254	5.104881
48	11.575987	-0.841191	1.293563
49	2.0024741	5.8761575	3.416269
50	7.226486	9.6423706	6.413726

Data simulated for Cobb Douglas production function with Additive error term

($y=\beta_0 X_1^{\beta_1}, X_2^{\beta_2}+u$) when N=100

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	13.46119	3.5576842	5.1387725
2	3.7832448	5.0460725	11.920859
3	4.9223332	6.0824528	7.4307608
4	7.7236504	13.89276	7.505435
5	12.675693	2.5861237	11.021413
6	4.1685796	11.766603	9.8079649
7	9.5799268	10.183739	2.6080082
8	12.141618	9.8239152	4.0136301
9	8.9416475	10.113653	6.4928319
10	11.675242	2.7871315	9.8211808
11	0.8904593	2.9653095	7.7390342
12	2.9294835	2.4974901	6.287206
13	3.3033239	10.13856	7.0942505
14	9.3679112	4.7083286	9.3541524
15	4.1827599	7.8841152	12.774903

16	10.072881	2.9697284	12.146803
17	3.2054623	10.087005	3.5121523
18	11.095271	9.175362	11.445432
19	11.090583	5.8380476	11.300655
20	5.9776546	10.760817	5.788605
21	11.207764	13.586498	3.9660972
22	14.343594	4.5721377	1.3315278
23	2.5453533	8.6111427	4.3636393
24	8.5463386	1.4530187	2.0385558
25	0.8630849	4.2233852	6.1247306
26	6.2639893	6.1517598	10.586457
27	7.4317559	0.1967795	10.442467
28	0.6095011	4.5730731	11.724108
29	4.0005323	9.1665428	8.3749306
31	10.672789	4.802343	11.698749
32	7.0640949	7.0234237	7.4100526
33	6.9489911	5.0802194	7.2539349
34	7.4091219	4.6074318	5.5259155
35	4.0151285	1.5065997	5.5281218
36	4.0530924	12.768616	6.1809365

37	11.671531	7.8828092	4.327848
38	8.6293696	10.558903	11.266126
39	10.351038	1.2506969	2.9270327
40	1.1835531	10.526714	8.3706197
41	10.458872	5.8304725	4.0426087
42	7.1083052	9.7223909	4.7433107
43	12.046991	10.334455	8.8278211
44	9.2765325	7.3730855	3.5359929
45	8.7418543	6.2499747	10.035962
46	6.3838151	6.833127	9.2284858
47	7.5683915	9.5952247	5.6319207
48	9.7727488	2.4468373	0.685704
49	0.0253483	7.2880802	2.6971028
50	8.1273791	5.9338518	8.4605721
51	7.0278352	8.8296492	10.59849
52	12.212368	7.4484471	4.2495654
53	6.1291244	11.64645	6.6542274
54	12.004737	5.3952046	14.018726
55	8.6642217	2.6786004	13.377752
56	4.917894	-0.2843	13.682352

57	1.1697896	0.259775	11.772418
58	0.7662318	5.6875169	7.0661017
59	5.7081128	4.330924	3.0681363
60	5.811473	6.6249571	2.1688492
61	8.3422624	6.7683919	6.4845448
62	5.8546678	11.174072	12.395601
63	11.009135	4.6304668	3.0325706
64	7.8642299	7.3926781	12.132214
65	8.5370468	6.1830223	4.9952545
66	6.4460219	9.3163543	13.933298
67	10.929216	5.9196633	7.3519894
68	3.6105215	4.2015113	13.646623
69	6.1916602	8.06386	6.6550506
70	9.8065672	1.3726672	8.1482076
71	0.736756	10.915984	4.1338367
72	13.264616	7.4754757	11.533602
73	4.6529579	12.482406	8.5847217
74	12.300049	6.061529	13.199145
75	5.3405127	4.5542731	13.882236
76	4.1805139	6.0090088	7.0837866

77	7.5074411	13.206078	10.404459
78	11.866655	12.835972	6.512291
79	12.199982	8.128818	2.7661574
80	5.8668918	11.842382	8.382069
81	9.840291	15.38321	10.397296
82	14.205973	4.315128	10.35798
83	5.7167873	8.9259082	8.6506953
84	8.1820177	6.6389738	6.4554362
85	7.7417165	3.292474	0.7166338
86	5.7428091	11.895414	11.449691
87	11.519203	2.3976288	10.839801
88	3.1656057	11.906647	4.0458759
89	10.032363	1.3466012	7.6227387
90	2.1841038	3.7585764	11.70553
91	5.6087305	1.3882182	11.592486
92	2.6116239	3.69674	-0.611651
93	4.3938205	1.3733977	5.5410925
94	3.1979861	5.6333241	8.0232405
95	6.5947126	13.949632	12.942234
96	14.216911	8.1000115	15.689688

97	11.829223	9.9625884	3.9347892
98	10.368958	6.1956049	7.0669086
99	5.5389914	5.4792422	6.6520863
100	6.4604678	10.288928	3.1206996

Data simulated for Cobb Douglas production function with Additive error term

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} + u$) when N=150

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	14.847245	4.5757855	3.2356375
2	3.2055972	5.463023	12.192481
3	5.8337199	6.9548841	8.4098209
4	10.275841	10.811576	6.9457162
5	13.164519	1.6039889	11.19819
6	3.214875	8.6004789	9.0729433
7	9.730188	13.549926	3.0160475
8	13.574627	5.5373617	3.8522904
9	7.1430996	6.2659067	7.9644322
10	7.972728	2.7676443	9.2867396
11	1.4335009	3.2397777	8.981218
12	2.7546776	2.8421344	9.604278
13	6.0371528	10.163445	9.5325432
14	11.339705	6.9191286	11.669377
15	7.1870369	10.936823	14.949836

16	12.496096	8.9138648	13.451368
17	6.7169286	6.1625517	4.7584015
18	8.3732055	10.544289	8.1085628
19	12.2971	5.7419697	14.155283
20	5.2707593	9.8320457	7.2035172
21	12.173385	15.179315	5.4048144
22	12.28729	3.69482	0.292803
23	3.6179433	10.347689	3.8958551
24	9.5685727	1.8439795	2.6095378
25	1.2864285	3.4360207	4.0761211
26	4.8254954	6.1960648	12.874969
27	5.0469912	1.070794	12.286928
28	0.152922	8.0225184	11.891883
29	5.760017	12.992436	6.4094696
31	10.964966	6.7793642	10.442685
32	5.839564	4.8568675	6.5801957
33	6.2824022	7.7832757	5.7344183
34	6.9847567	7.9754883	4.0029952
35	9.0639648	1.4133444	6.5700223
36	3.7749694	12.84152	4.6479287

37	12.429182	7.323705	5.7393803
38	8.4539485	12.998945	12.254151
39	11.093361	1.1305196	4.5589064
40	1.9529994	10.653085	11.140164
41	12.398794	5.0512568	4.1432483
42	6.4231391	11.556712	4.5215456
43	12.215127	8.7948761	11.852149
44	11.017061	6.8749884	4.6960099
45	7.9325194	9.5002672	10.316675
46	9.9250648	5.3709386	10.767876
47	9.024041	9.8636985	4.1596153
48	10.271954	1.7691	2.0769761
49	0.4594713	6.4289784	3.1866256
50	7.4028476	9.4234154	4.9352113
51	9.2017717	9.8833718	10.05762
52	9.5376377	6.3350059	4.1632623
53	5.6714639	11.363567	5.6445241
54	11.786457	7.8407425	12.815926
55	7.330182	2.9196035	11.787656
56	4.4997892	-1.303331	12.664938

57	2.7783283	4.1456782	13.701774
58	1.5459349	5.4118207	7.0955066
59	5.9889753	6.0581603	6.9258434
60	6.2546204	7.8686485	2.3475759
61	8.1115901	6.7586717	2.7422687
62	7.5712198	13.545817	12.177352
63	13.836889	3.5626387	1.1840523
64	2.6816294	6.6516979	12.839582
65	7.1931481	4.1370891	2.2370159
66	7.3298914	10.779455	8.1431549
67	11.276899	5.0221548	6.2822136
68	2.5172863	6.7796628	10.601863
69	7.6099122	10.43919	9.1282453
70	12.725126	2.2453951	8.7794659
71	0.1941893	10.678691	3.4612911
72	12.801915	2.6036113	10.78576
73	4.3526839	8.9437792	13.852786
74	10.147916	3.9791336	11.22514
75	4.949386	4.5425738	14.236063
76	6.9116988	6.4179851	7.7364216

77	6.4162882	13.638567	12.115484
78	12.164921	9.6989275	4.2844645
79	10.018496	1.8735026	3.0696456
80	4.6581134	10.239578	11.297442
81	12.852466	14.280936	11.286525
82	12.642729	4.7493806	9.6129725
83	6.2422135	4.4791323	9.6823426
84	8.1469632	4.76242	7.5977588
85	7.4824972	5.5510114	-1.055946
86	4.4087761	9.6130954	12.724256
87	10.777644	3.2128859	12.912736
88	5.3503297	10.806399	4.057143
89	10.356594	0.8164903	7.0523079
90	1.5859733	4.6755744	10.454837
91	3.843624	2.2225305	12.829477
92	3.0557063	3.9803211	0.2824201
93	2.7159798	2.085735	7.3610286
94	2.8599367	10.10655	8.484392
95	10.157475	8.1430561	10.136383
96	7.1055365	11.21032	14.023592

97	9.523792	6.961343	5.8824839
98	11.344571	5.6922433	5.7514128
99	5.7910742	4.2343103	7.3962152
100	5.0919824	10.468384	5.2640747
101	11.214556	10.450783	13.613381
102	8.1020464	9.8397946	9.5384223
103	10.53786	6.2051221	10.899235
105	6.5504951	3.9044557	5.2272769
106	6.2796008	13.22239	10.405167
107	14.853996	9.3016344	10.690546
108	9.2109917	3.5414487	7.3023259
109	2.8086393	-0.601974	-0.671156
110	2.7452787	7.2902513	6.8559222
111	6.3294317	11.84406	2.7998687
112	12.202353	8.9019699	4.8301111
113	9.9547695	13.849192	8.6994319
114	13.888252	8.6117866	3.59378
115	8.5486625	4.3075637	3.5225776
116	6.6167206	7.56868	3.6374845
117	7.7855428	4.2269473	14.32621

118	5.3147064	0.4811495	10.256524
119	-1.603099	5.9918427	7.917779
120	6.8907555	0.6469971	8.9367261
121	0.3715695	6.2649194	5.1720918
122	6.6123644	6.1004228	8.3192406
123	8.1771109	8.2851585	6.2626057
124	11.523428	5.5137075	12.445547
125	5.4801309	5.5910579	12.660512
126	5.8117161	4.8222075	6.486351
127	2.2848629	9.3010188	7.6980432
128	10.473262	6.428141	7.9806963
129	6.1669825	3.9603575	8.1765109
130	8.2351928	3.330045	6.0174004
131	2.7515979	2.1123803	11.68172
132	2.18197	9.8529658	4.9810647
133	10.239197	4.8492998	5.1648311
134	4.3608596	0.9067061	4.3916881
135	2.1594726	2.0498153	1.9908326
136	1.3762797	7.5727265	5.3924877
137	9.2557505	7.0633338	11.730061

138	10.65453	5.3603332	7.4431474
139	6.9986578	6.4361451	4.6030326
140	6.42543	6.7390844	3.7670341
141	6.6313894	10.30755	11.532682
142	10.701199	7.032025	8.6107906
143	7.3580741	7.5151478	9.5173825
144	10.313455	6.7028208	6.033268
145	5.3739857	4.3634156	7.6636228
146	4.2152024	2.1195365	13.106558
147	4.082637	11.634487	12.31713
148	10.908517	6.9404957	6.1799649
149	6.5215478	2.7352252	10.278628
150	2.8026774	8.4880007	14.206631

Data simulated for Cobb Douglas production function with Additive error term

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} + u$) when N=250

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	11.955613	4.4429762	3.221818
2	5.1559407	6.4840104	12.004102
3	6.8285578	7.7443784	7.7025689
4	10.217185	12.186199	5.9741445
5	13.847779	3.7676699	10.660271
6	3.5325967	9.8868106	10.352296
7	9.7357338	8.377435	0.566381
8	12.198581	9.561267	5.5034672
9	8.614278	8.489162	10.267083
10	8.8299963	0.7826231	8.875112
11	1.7458699	4.1307224	11.079624
12	4.1045028	2.1142112	10.272251
13	1.7572247	7.9982117	8.6540703
14	10.45972	6.9017583	9.0730656
15	7.1188417	12.764915	13.869009
16	13.897166	5.8531224	11.542247

17	4.4362279	6.0083978	1.7767712
18	6.9089895	12.886596	9.2639344
19	12.566473	5.376763	12.889
20	5.7144082	7.4464118	6.3219597
21	9.5905167	7.8409022	3.1828702
22	11.040445	2.779799	0.8111281
23	2.4363752	7.5633852	3.5366336
24	7.8228888	4.2738893	4.4535244
25	1.6623332	3.7477736	3.8848403
26	4.6967873	5.7104694	11.590209
27	5.3982411	-2.053485	7.8695501
28	1.6312948	5.2283717	9.806559
29	4.8291732	10.093618	8.9471168
31	10.274407	6.3174513	9.3565915
32	6.7226569	4.1971125	5.8229762
33	3.6001124	7.9207985	3.6300874
34	9.4388759	9.0690627	0.4732231
35	8.5879308	2.0118723	8.2762265
36	3.4695631	6.3090313	6.3559617
37	9.819852	6.0301922	5.1652785

38	7.7131564	9.0476946	13.949133
39	9.0880177	2.3381509	5.5049052
40	1.7726535	7.9147888	11.593852
41	8.1804286	5.9918037	4.2416755
42	6.2565597	8.9411877	6.8657238
43	11.785081	7.1320877	11.302905
44	5.8334033	12.23438	2.7455856
45	11.935234	5.6122852	10.117399
46	7.7372124	9.1697817	8.2795409
47	8.4432838	11.362976	6.1849361
48	10.885288	1.0866026	1.5679859
49	0.8253707	5.9899632	4.5970553
50	6.1929156	6.1849611	6.3338694
51	7.3412219	10.316682	12.135281
52	10.73461	8.021549	2.807503
53	9.0104073	7.1393069	5.3889284
54	9.9527911	8.1657881	12.962265
55	8.6444787	3.237641	13.85497
56	3.7693645	2.7899647	11.334279
57	0.1173028	3.3881681	16.684515

58	1.7106584	3.844985	7.6882383
59	4.4434908	8.5385536	5.376658
60	8.0844565	5.0279804	3.946307
61	6.6534003	5.8829799	5.6586352
62	4.5107821	10.384637	12.40239
63	9.5515422	3.6827401	2.1795947
64	3.0401986	6.7366126	12.579633
65	7.8493538	4.5565977	0.9781756
66	5.9435222	7.7912984	8.705196
67	10.082248	4.9194719	6.69974
68	3.938079	4.2144976	8.8198702
69	6.4688696	9.4691949	9.2413168
70	9.757553	0.3141832	8.2665388
71	0.3080632	12.605265	3.025371
72	12.916585	6.5302942	9.2979648
73	5.106807	11.651324	10.834773
74	10.022394	5.6416203	13.289958
75	4.5676648	3.7130902	12.159792
76	6.9884014	7.9401161	11.610869
77	9.3730805	13.864478	9.4653456

78	14.524506	12.266357	3.62064
79	12.571745	6.289772	4.2231498
80	5.1838745	8.3239203	12.929843
81	10.160415	14.381584	11.194954
82	15.031072	6.739789	10.358983
83	8.2099858	7.0837553	9.8071832
84	9.6713878	5.1053637	8.3655687
85	6.2206853	6.6020653	-0.417449
86	4.7544227	6.6607813	11.413698
87	8.5062556	3.6460996	14.740408
88	3.3748469	10.323098	5.4570782
89	10.002425	1.5598215	7.8104167
90	1.6940017	3.4507497	11.260918
91	3.0335784	2.0650433	9.6368326
92	2.2710288	4.8773037	0.6137367
93	5.2125084	-0.234087	7.7510017
94	1.0810023	9.1560323	8.4975172
95	9.4070626	10.710686	12.307251
96	11.429089	8.8126314	12.902773
97	12.838996	8.0013839	5.4401151

98	10.181155	7.2666703	6.3030129
99	7.1093572	4.5367655	5.9188012
100	5.5478902	10.979758	4.8038415
101	9.8458115	10.835849	11.535838
102	9.2581228	10.343349	9.9744409
103	10.678881	4.0678513	9.5132136
105	4.5566883	4.7816458	5.7479384
106	3.3045949	12.324865	12.247108
107	13.157326	11.486085	6.979016
108	10.706565	3.4958223	6.5385995
109	3.2617423	1.1300687	-0.773551
110	3.3555024	4.5291318	9.4747245
111	5.3348754	11.719359	2.5216657
112	12.723686	7.9404434	7.4428544
113	9.3958458	6.8315614	7.5048182
114	10.803195	10.404795	2.456859
115	10.298989	5.0895409	1.4255101
116	7.1226622	5.0150352	3.2005949
117	6.2528537	2.2495819	14.331911
118	1.1034281	0.9042856	10.707823

119	0.4870793	7.2204212	11.691678
120	5.9548797	2.3780704	10.717559
121	1.5328414	6.1823023	6.7408738
122	8.1137353	7.2468742	7.344571
123	7.4283768	12.228704	11.414045
124	13.262448	7.3847935	8.8484195
125	6.9671405	6.8733442	8.520822
126	7.7089856	2.2424381	4.1955766
127	3.6911676	7.4491871	9.9234792
128	9.8783077	6.1867584	9.3851891
129	8.0245561	6.7118652	10.383614
130	6.0586391	3.3930906	6.1058365
131	2.9868752	2.6483093	12.150276
132	2.5461966	9.9340681	6.6775358
133	8.556794	8.459936	4.4653668
134	7.2529444	2.44359	6.9581015
135	0.7383892	-0.239066	5.4906957
136	-0.148973	10.198334	4.3700493
137	10.87361	11.63338	12.83775
138	11.86289	7.0477281	9.4475932

139	8.7434433	6.3891265	2.8484604
140	11.530057	8.4542587	1.0593316
141	8.2713907	7.6719835	9.6977064
142	11.37068	7.2769261	6.9117441
143	11.649041	8.5784235	6.2092657
144	8.6977976	7.0555097	6.2825462
145	7.209656	5.1625131	10.418396
146	3.4175522	4.1774695	13.386818
147	3.7385149	7.6647414	12.750691
148	6.9819949	5.02683	3.988905
149	4.9546968	4.4702709	7.042822
150	2.8002106	9.1796081	12.214175
151	10.144461	0.8856043	8.5492913
152	1.92825	9.8428556	4.9034663
153	9.560195	9.9790347	11.023177
154	7.4970604	8.2189005	4.6456481
155	7.7431111	4.8576864	13.397831
156	6.246121	6.8091513	6.499111
157	9.1477935	7.5357593	10.784377
158	6.399741	2.64168	5.9912488

159	3.0534213	6.5738753	5.0573026
160	9.2797409	1.0962384	3.3877575
161	1.7202024	3.3686327	8.9346393
162	3.9563864	2.9815076	3.3257511
163	4.5615617	4.4493165	13.953503
164	5.2891264	11.703634	8.9268081
165	12.613354	7.97077	4.388348
166	5.4314342	10.678101	14.091712
167	11.388769	11.969889	12.061164
168	13.384439	6.3587877	11.000281
169	7.6707396	2.112573	10.913948
170	2.0686046	4.1841547	3.794324
171	5.1620489	8.9957367	9.005651
172	11.930733	7.13824	8.7124204
173	5.8020672	9.8180037	8.8114392
174	7.8381299	10.523898	3.0139513
175	11.689665	11.948774	8.428166
176	12.195199	2.6114997	13.304654
177	5.4184951	5.649432	4.9159335
178	5.2421069	12.951512	7.8159087

179	12.580893	4.5355044	8.5032093
180	9.1976888	10.515935	10.023366
181	9.7094583	8.5751038	6.5883538
182	10.118291	12.656728	8.4363971
183	12.76966	2.8811475	4.1629706
184	4.6130992	2.5034687	4.1361847
185	0.6639975	9.5510961	5.2167256
186	11.445195	7.1183339	4.9349261
187	7.9602335	9.0866641	5.1720301
188	8.1420103	3.0088724	6.9928961
189	5.4628446	11.898552	13.418604
190	10.570041	6.1911928	3.378755
191	5.0445814	11.657635	7.3843626
192	12.685355	9.2275599	6.6448788
193	11.23203	5.4120718	8.6004175
194	8.5676928	6.7916156	12.79216
195	6.8712674	1.6391536	8.274868
196	2.7131993	13.547461	8.4447998
197	13.523234	3.8985321	10.065806
198	3.7664361	4.8538224	6.687484

199	6.9185557	0.3222043	11.240928
200	-0.240937	12.253445	10.75829
201	11.647409	3.7582536	7.8921326
202	2.357046	4.7279828	7.5432796
203	7.1875705	5.5067462	12.241965
204	5.5921215	2.6392841	10.628189
205	2.7608296	6.0357722	6.9576316
206	5.8353581	1.6729618	7.3641161
207	2.6903034	0.3455979	13.111699
208	0.5493757	7.6954814	6.7695899
209	8.8649403	8.5226063	11.843581
210	6.9741643	0.2733234	10.160134
211	2.4829247	2.9789619	16.011252
212	4.9522431	6.9787067	1.6900775
213	8.7117091	9.6086423	9.377664
214	12.935579	3.7948628	9.0930483
215	0.1372196	4.6164176	8.6884047
216	5.5940863	12.7345	3.91371
217	12.122861	9.5807294	3.5599299
218	10.163604	5.4154976	14.130586

219	6.9121705	8.8093984	3.42472
220	9.2795487	10.6258	4.0287103
221	11.652231	13.871915	11.163996
222	12.79697	6.3581632	13.033323
223	5.1811449	3.0414463	7.0314229
224	5.6947149	1.3249663	10.236417
225	3.0037516	4.9477449	5.0833632
226	5.2310614	7.2453049	7.1628853
227	8.0662407	10.45753	7.7670306
228	11.090026	4.843417	10.128478
229	5.9202025	4.6621271	14.7752
230	6.5054625	1.0358475	2.1274168
231	0.3392033	7.142878	2.8574367
232	5.5483007	12.061808	6.0166024
233	10.078819	5.1235276	6.2351719
234	5.7127271	0.5175327	9.8149015
235	2.3387462	5.4806626	10.40701
236	5.6441364	9.4470219	11.783149
237	11.306881	8.1323917	10.881732
238	6.8434976	11.801206	8.7608026

238	9.9638773	7.5312109	11.93949
240	10.329606	1.4607117	5.3330961
241	2.0317246	4.85876	7.7039948
242	7.4587874	5.4331541	8.3099422
243	6.5037515	1.3369815	4.1673243
244	2.1164518	11.333884	8.0922157
245	9.0727932	9.8975376	9.7418584
246	11.463955	1.9222446	4.2799361
247	1.7400358	9.4674085	10.811973
248	6.258933	2.0554017	8.2886246
249	3.4012725	6.2310809	5.226234
250	5.6118115	1.0278472	13.303665

Data simulated for Cobb Douglas production function with Additive error term

($y = \beta_0 X_1^{\beta_1}, X_2^{\beta_2} + u$) when N=500

	$X_1^{0.85}, X_2^{0.15}$	$X_1^{0.90}, X_2^{0.20}$	$X_1^{0.65}, X_2^{0.20}$
S/NO	y_1	y_2	y_3
1	8.2265999	4.1183305	1.7809383
2	4.9287276	5.4213729	9.2125252
3	7.8165196	7.1724638	8.0406144
4	7.0121708	12.682137	6.3462621
5	13.194004	4.0473878	10.068157
6	3.8343672	15.130942	10.186626
7	13.659995	10.505015	2.4321244
8	15.026851	9.7501816	5.5979113
9	11.549798	10.507584	9.982531
10	9.1671	-0.034842	9.5637239
11	0.6550667	0.9939155	9.3426894
12	4.7515459	1.7691265	4.9513119
13	4.0716777	3.1194057	5.7764411
14	6.0161454	5.9971656	8.4628706
15	5.7364868	10.685623	11.426162

16	11.963178	5.9585027	10.837229
17	6.3713587	9.7741265	4.2153828
18	12.445933	13.547925	11.319134
19	12.734128	4.680895	8.1983531
20	5.627128	12.120746	7.8420892
21	10.295669	12.859276	5.3112871
22	12.788695	3.7594919	1.0586269
23	3.8175645	7.378794	3.221526
24	8.6658921	1.7451215	3.4823307
25	1.3906192	3.737203	4.398281
26	5.7887649	4.5911705	11.116296
27	6.6935829	0.151777	9.7006147
28	1.6899661	7.1554996	12.290436
29	5.3754518	13.795084	9.7901768
31	12.988807	5.8225244	8.8975905
32	4.591621	5.8457971	6.1652516
33	8.9253601	5.8438498	6.4774918
34	4.7394047	6.4686789	2.5537602
35	8.4638626	2.1057047	8.9501415
36	1.9745588	9.8797349	4.7614258

37	9.1626105	4.7996554	6.7798719
38	3.7739019	10.297855	12.436414
39	10.378281	2.9090207	4.1164823
40	1.9353124	8.5023871	9.9999504
41	6.8443359	5.8240468	4.8508486
42	6.2758756	10.026788	5.1489545
43	10.699633	10.401376	11.346643
44	8.4510145	11.566283	5.0673264
45	12.328132	7.0645814	10.48484
46	6.4600616	7.4375918	7.3283015
47	6.6319545	7.8052554	4.559014
48	9.437292	-0.191102	2.425282
49	0.08914	6.8867967	2.2125812
50	7.6603476	7.8968869	5.9001618
51	10.68604	10.521453	10.999659
52	11.967085	8.0002114	5.2252441
53	7.177015	11.502323	5.7819946
54	10.98896	1.7806508	12.693601
55	4.1418781	3.749514	11.660386
56	4.2737255	2.8615008	11.377997

57	1.2218285	0.8071806	14.659361
58	2.3226826	0.9858077	9.7806877
59	4.4398858	5.3690087	5.6456481
60	5.4904038	9.4470667	1.7942898
61	10.891004	5.92055	4.9621456
62	7.5449341	11.480863	9.9439572
63	13.860783	3.302	2.5499453
64	4.875804	6.7065512	10.723802
65	7.8818649	4.6413841	5.1512216
66	5.4754523	9.3883395	6.3120201
67	9.821664	5.2187623	4.2930853
68	3.264324	5.5686173	11.362467
69	6.2410004	10.650636	8.4234199
70	10.272139	0.6295401	8.2059206
71	1.9250063	14.091569	2.2060584
72	13.109165	2.9259673	9.2725321
73	3.9798978	11.6136	10.661175
74	10.365424	5.194187	14.336752
75	5.0180564	3.0172567	15.699648
76	3.5453037	8.2581487	9.3117399

77	6.5037336	12.88792	7.725521
78	10.950715	12.710337	4.6681566
79	13.574128	7.3980057	5.4512802
80	5.2129525	9.5922416	11.146653
81	11.039508	12.446905	12.636539
82	13.556871	4.361967	11.86949
83	5.7314353	11.108586	12.645479
84	10.713168	7.6113007	6.3332307
85	6.1356576	4.4750005	-0.479518
86	2.8941681	11.018509	12.070869
87	9.8633057	3.8703926	11.233732
88	4.1004957	6.4624013	3.7475928
89	7.5318287	2.4521496	7.6368444
90	0.8311154	4.5814138	10.713514
91	5.3258959	1.6757725	10.86106
92	0.9921911	3.3905139	-0.120708
93	3.6740319	0.7493146	4.7807903
94	0.805704	10.971274	6.8201635
95	8.9770384	11.18964	5.423685
96	12.861405	9.4274633	12.799804

97	10.290705	11.987016	5.7827968
98	13.114156	5.5628855	6.6308114
99	7.2505627	5.4065336	5.9599965
100	8.0273667	11.139404	4.2768533
101	11.499591	7.3720372	12.684046
102	6.9936983	9.4921448	8.5673235
103	10.795041	7.2952331	11.170092
105	6.5736058	3.6792635	3.6537127
106	4.7281315	14.609558	13.16359
107	15.351564	9.7606674	7.0736881
108	10.596643	2.5768829	4.6551499
109	4.6486742	0.4340839	-0.410175
110	1.6731658	6.1851807	5.671314
111	8.5506518	15.832897	2.5016057
112	12.156862	6.2641139	7.2552527
113	5.0528349	14.645571	7.4014789
114	13.255942	9.4926416	2.3105609
115	10.720663	4.8453601	4.0952465
116	4.5754483	4.8218813	5.1388402
117	6.307259	5.1917491	11.331196

118	2.8976236	-0.170974	7.7171596
119	-0.155496	7.3238374	11.68558
120	6.4297164	-0.226678	12.244899
121	3.0186833	6.3611695	3.3319775
122	6.9671107	5.4210371	8.1125878
123	7.2170667	14.163724	10.4698
124	14.546323	4.0641352	12.435521
125	6.0175845	8.6890705	12.022418
126	7.8879722	3.8449573	4.6292438
127	3.3252479	9.9897818	10.78249
128	8.4708022	5.5978051	5.4038611
129	5.9791591	6.9582073	6.3336385
130	8.2135455	3.0803571	8.3285043
131	2.7759898	3.120764	10.588456
132	3.8557419	10.654039	4.5787492
133	8.1376531	7.6100852	3.8722309
134	9.844113	0.713699	5.988563
135	2.0573926	-0.140742	6.6572646
136	0.4689298	9.7252741	5.0822356
137	11.570201	10.398419	9.2600968

138	11.991836	6.7971063	8.3508661
139	7.86031	6.3156227	2.8365727
140	7.5758585	5.3343148	3.1097992
141	6.2234864	12.764086	9.9945037
142	14.858222	8.0763949	8.5709323
143	9.4010084	7.9961711	8.9423777
144	10.927004	7.1299044	6.3945078
145	8.7852486	4.7905247	11.930381
146	3.7542522	2.5610174	11.866832
147	4.1969835	10.740615	11.119189
148	10.767396	3.8580492	3.3231297
149	7.3874831	2.4062545	10.890349
150	1.9760211	9.3446801	8.9013127
151	11.474508	1.5055754	9.7211309
152	2.0829396	12.746962	5.9088494
153	11.101741	9.010935	10.243305
154	8.35908	7.7399978	3.0472768
155	10.857043	6.3277283	9.5672441
156	5.7096966	4.6715089	5.8074364
157	6.6914188	7.4375476	9.6540095

158	7.4914135	-0.928844	4.8887021
159	2.3015579	6.0996186	4.3100457
160	7.2711086	1.8247877	1.2404519
161	2.3176574	3.3880292	6.9084534
162	3.8560029	4.3248354	3.2664848
163	3.7976647	4.3049507	8.2113201
164	3.1898955	13.955956	10.248195
165	12.770134	3.266775	5.3161724
166	5.4415021	9.9419932	11.443346
167	9.8662531	9.1139308	11.025662
168	12.553936	5.7648851	13.235263
169	7.7122354	-0.101	11.508973
170	1.4165458	4.7541185	2.9278319
171	4.9232927	9.2574182	12.38646
172	11.154943	7.4025222	9.0807824
173	5.0307258	8.2361331	7.9616747
174	8.8036896	12.448573	3.6554195
175	10.58817	12.032691	9.3693976
176	11.802767	4.2160392	10.788696
177	5.4271115	6.0494597	2.806489

178	6.3211392	9.9841168	10.146502
179	11.04617	7.1879686	11.258339
180	8.5907634	6.8048252	11.607517
181	8.6499588	7.7056618	3.7981907
182	6.8759705	9.5848399	9.670978
183	12.762811	5.8046642	4.8301135
184	6.4200713	2.0203844	6.0823886
185	1.736091	9.3477684	5.1143089
186	9.6376857	7.2260717	4.0938146
187	7.2040225	10.958178	7.4766148
188	12.588111	2.4962714	5.9674881
189	2.1160184	9.00848	12.622637
190	9.9007753	10.422357	3.344859
191	10.818876	7.411116	6.0516622
192	8.3239649	7.5381916	7.3422131
193	9.1077401	7.5862171	8.6621339
194	8.0205804	5.5223391	11.205055
195	5.4237314	-0.137958	5.875665
196	1.628776	11.891111	9.8582426
197	14.186746	4.5211363	12.406364

198	4.578841	8.4583052	9.500104
199	8.1041521	-0.4742	12.680374
200	3.8579207	9.5565284	13.46936
201	9.3151986	5.971552	8.0077377
202	6.7683666	10.058054	7.1840439
203	11.47093	3.8044128	5.8122033
204	5.2163618	5.710925	12.94065
205	3.7450419	7.7297623	7.3017037
206	9.0851637	4.2405844	10.008134
207	2.9911073	1.8323911	12.133504
208	2.3014582	3.5805763	8.154127
209	5.5556263	6.0065976	10.394393
210	8.2454891	1.6725645	10.339419
211	2.4969318	5.7154739	13.246343
212	4.3875383	12.155186	1.4960358
213	10.462941	15.142606	11.06595
214	14.291401	2.9226215	9.1912372
215	1.2316878	7.4628823	4.6154751
216	6.4867368	12.389296	0.969268
217	13.821614	10.293995	1.6041918

218	12.043473	1.9844337	14.458308
219	5.843736	4.6443628	5.4031713
220	5.771778	11.44632	1.8181275
221	10.737427	12.23025	9.2301053
222	14.173585	6.4832113	6.7016678
223	5.1181715	2.0342681	6.6403783
224	2.0270123	1.9067425	10.170166
225	3.4987636	7.0493559	7.4098051
226	4.9976895	7.5572037	10.416482
227	7.5712725	9.7155083	4.0830185
228	12.04293	8.0072313	12.858746
229	7.5505857	2.034615	9.7913199
230	5.1019599	1.7127606	0.093551
231	-0.180663	6.2860912	7.5107181
232	4.282728	6.8384843	8.9283279
233	9.0283032	4.9476869	8.3182212
234	5.6894818	3.0778827	7.4041748
235	2.0076459	4.7900434	8.5905769
236	5.7138366	4.4800613	9.89576
237	7.4816884	6.4357861	10.475619

238	5.3224478	7.9742455	10.741082
238	10.079509	9.2662485	8.8938879
240	8.0115765	8.3066414	5.9795847
241	8.7036643	5.4918912	5.1288561
242	4.8515862	4.4069816	9.1677051
243	5.3385251	1.9457434	2.7549843
244	2.9102192	6.772732	12.152327
245	8.3356454	13.767435	8.8532041
246	12.719304	2.6930678	3.7282009
247	2.9134838	3.985477	11.263333
248	6.7239312	1.7030622	5.2332602
249	2.4760373	7.0587946	6.9957505
250	6.8526998	1.538055	12.516612
251	3.070819	6.6944428	11.905629
252	6.8354874	14.926779	10.135084
253	13.366358	8.7559629	4.6933273
254	11.865954	12.958543	2.9985576
255	14.010463	5.5074723	9.4191245
256	6.934539	6.9853252	8.4683716
257	7.6917479	5.3518117	4.5024974

258	7.4575993	-0.569878	3.6265927
259	2.1621819	5.6383179	2.0438489
260	5.0480195	6.9555943	7.4762455
261	6.3512112	6.9412322	14.814081
262	6.2092028	11.359517	13.202615
263	12.354112	7.9609418	9.9821019
264	9.4109801	4.3100913	11.987887
265	6.2362483	3.3241729	8.3298173
266	5.909527	8.6978328	5.5757411
267	10.700011	7.9045322	12.368598
268	10.344493	3.7597322	9.6695625
269	1.9034572	1.7768525	1.4193319
270	-0.131185	12.207186	4.3118897
271	10.147305	9.646485	9.5940578
272	10.054669	1.722671	4.0672296
273	3.1754207	0.7229452	1.1693667
274	-1.44904	0.2971155	12.073698
275	3.3988794	3.7069234	13.552482
276	3.358201	4.4477106	11.200307
277	3.2563135	4.553443	1.6790194

278	5.4229143	1.8890446	4.8639133
279	1.942108	3.2545131	2.7743756
280	2.0650972	3.1865177	3.3930041
281	4.4000813	5.2487002	2.9247625
282	8.2966223	12.988362	5.3740085
283	12.708123	-0.174927	4.8173841
284	1.4413429	9.0285551	0.1135611
285	6.5666238	11.719063	0.3168136
286	13.894156	5.707276	4.9185232
287	6.9210639	1.8376761	2.9816289
288	0.7550849	0.432703	6.7634727
289	2.3235025	5.3081386	9.1025587
290	6.2459823	2.0768803	11.949909
291	4.7074794	2.7511487	12.743369
292	1.2785208	1.2932673	10.077092
293	3.6491148	2.7014817	8.735067
294	4.6516472	2.7483135	10.466873
295	2.6581492	14.751593	5.0196948
296	14.932104	5.2510571	4.5711543
297	4.7881564	6.9143594	4.1202806

299	7.7935331	8.639744	12.428733
300	10.9591	1.3947227	12.30187
301	0.6174593	0.7083751	13.809434
302	1.0005524	-1.619073	8.3819877
303	0.4352337	12.540004	5.6160641
304	10.606697	9.110393	4.3591218
305	8.810861	-0.908032	7.1352161
306	2.601488	8.8798287	15.665037
307	7.261041	7.0404872	11.66665
308	6.9344617	4.1895248	10.844609
309	4.232078	13.585525	9.770268
310	14.553492	2.1216319	12.750504
311	3.4571208	13.317881	8.7584052
312	13.671268	1.8087253	11.774839
313	0.8429266	1.5742704	13.782397
314	5.3472976	2.4450402	8.1983396
315	2.5668711	3.5535793	2.3073699
316	1.9183442	11.117673	9.3740462
317	12.248593	8.644347	10.698747
318	12.788494	3.5409622	5.1207339

319	5.0339287	9.0384103	13.317525
320	7.1052059	2.3709813	3.1546783
321	0.444907	4.0787025	13.448542
322	6.1605049	9.3236731	13.359623
323	10.466058	9.6811903	8.9750251
324	7.7525722	8.2390813	5.6732828
325	9.7549264	5.6406856	4.931122
326	4.8091032	6.5694995	12.453667
327	6.264602	14.445345	3.6665122
328	13.867387	12.853412	10.995205
329	14.174399	7.5526788	8.9000011
340	7.1595014	7.9207644	11.907426
341	9.5721694	12.189496	12.937982
342	11.039834	2.2045312	13.243418
343	4.3888875	11.006919	13.286303
344	10.737986	13.249671	9.2455395
345	13.990141	6.6854342	10.108275
346	5.025959	7.3610045	12.890302
347	8.3869048	11.267018	5.174173
348	12.207613	1.4507868	3.5788832

349	0.8269354	6.5725389	13.396393
350	7.0327816	6.5422574	12.919967
351	3.8282635	3.0557896	3.4288892
352	3.579534	8.0939399	12.087864
353	10.638781	11.75151	2.2963685
354	12.031169	16.632011	9.3230873
355	15.015882	0.0371269	5.2831878
356	3.4920673	9.5475944	8.3120526
357	8.3080717	5.23825	10.022796
358	7.5024282	8.6301858	11.482455
359	11.534404	4.0569053	7.9777558
360	4.5408411	4.9064158	4.7976317
361	4.0041288	3.1175411	-0.36994
362	2.3852865	3.7658216	5.2889322
363	6.8108687	9.4702889	10.293942
364	8.1883123	8.2024132	11.0934
365	8.5589771	1.8471099	3.7220426
366	0.2282042	5.6120742	11.750988
367	6.8798996	4.4480934	13.225184
368	7.4718005	10.545212	1.3955922

369	11.152103	1.5346689	5.2791567
370	2.2887096	9.1156059	3.9005054
371	9.9923762	0.5293501	11.710802
372	3.1246692	9.7840919	4.015572
373	9.6120029	5.4004664	15.794582
374	6.4845128	1.7393492	10.492349
375	0.8114259	5.263455	3.5032758
376	7.7908906	11.08324	3.8684961
377	12.048607	2.1016444	10.345503
378	4.8602461	5.7004665	11.61347
379	7.1607659	12.023856	2.5916868
380	9.6595335	1.1492847	11.500268
381	1.9410439	8.2606967	8.8649167
382	11.069032	10.415393	4.7734936
383	10.562284	4.3782864	6.9722325
384	3.3616688	7.7622792	6.4163133
385	8.7639475	9.2285625	2.0350565
386	8.9691696	12.841507	3.2430734
387	12.253297	2.5280249	4.8077393
388	4.6900931	2.8480187	9.336443

389	3.4998606	2.992377	11.868034
390	3.9119265	4.436602	14.121626
391	4.0367678	12.15787	5.1361168
392	9.9883478	3.2963448	6.2363253
393	3.1266364	7.5599873	0.4198786
394	8.9665635	4.6858256	3.3965855
395	6.1845831	9.495901	7.1667877
396	9.1719016	10.123969	6.2547033
397	10.645733	14.58396	5.5365552
398	14.892103	4.6016519	12.752715
399	3.6370203	5.7199995	11.54177
400	4.0094518	10.748702	9.0367615
401	11.233059	13.400205	8.0587308
402	10.716116	1.032117	14.62176
403	0.7336185	11.89047	5.7820093
404	11.522102	0.9324336	2.3819403
405	0.3463045	1.2410201	12.534599
406	1.525382	9.9560925	8.4682132
407	8.5014601	2.7620207	9.400608
408	3.5051416	2.9107047	4.7817715

409	4.1069464	3.7812376	5.7835265
410	4.8205119	8.3640229	3.9810736
411	10.002634	3.9368664	12.014635
412	3.3766525	0.2221509	6.5392269
413	2.1538344	1.3385317	6.3988347
414	1.6555827	8.390436	4.9256079
415	9.8639607	3.3105896	10.659239
416	3.5237629	11.356229	6.8364151
417	11.829123	8.6284899	9.7516866
418	12.459918	11.223283	13.420598
419	11.081879	5.3958388	4.5648857
420	6.7592905	4.1533032	1.6569583
421	2.5878713	8.4343539	5.700765
422	8.2812143	0.8706118	6.8282306
423	2.8174619	6.6210951	8.814188
424	9.1792197	13.194748	10.250198
425	14.44457	5.2044977	12.702844
426	6.2453788	9.2655255	1.4077523
427	11.269815	7.2330554	12.004819
428	8.036406	6.6804432	12.861305

429	6.6614519	5.8818722	9.0446345
430	8.5035776	5.3016106	1.253278
431	7.1513891	5.5553584	10.318427
432	5.0250594	10.114551	5.5114052
433	10.551358	14.225387	8.9657095
434	11.028824	6.3986184	7.996958
435	7.3816797	10.523868	7.7814761
436	10.305838	10.679646	4.2911646
437	9.0422401	0.2305788	12.943348
438	1.7500838	6.0418113	11.944234
439	5.0813452	5.4809513	9.3367562
440	5.5693607	0.5274588	4.5299935
441	2.8324738	10.799595	4.169897
442	12.900516	12.085122	11.727938
443	13.674348	11.621934	7.258521
445	12.342392	13.613363	6.827672
446	11.340511	2.306946	3.6997362
447	2.0820213	4.5347255	11.022792
448	4.6120873	4.2134292	9.8578811
449	7.0841068	2.6520272	15.127926

450	2.3516595	13.780582	6.1092307
451	11.979729	11.329382	13.016054
452	12.277622	-0.454011	11.682883
453	0.4299209	7.182338	11.74522
454	7.2118299	10.095122	12.920468
455	11.923314	0.5838663	9.3387296
456	1.670952	5.5787362	9.9493535
457	4.8413981	5.335859	14.362205
458	4.6909965	8.2802188	9.5711264
459	7.6111231	2.9364783	3.2293034
460	3.5514243	6.6165944	10.909249
461	6.0465202	6.9599152	12.211154
462	4.8537437	15.702892	13.187524
463	14.371305	1.7515949	5.2170959
464	0.4475535	1.7911952	4.241649
465	1.3652011	10.234152	11.193059
466	8.7505038	4.9823143	5.1388563
467	7.0058313	6.1391697	7.9561224
468	5.5211771	3.1818846	6.5881177
469	3.9124768	0.845338	6.6685224

470	0.7439687	8.0425169	12.950391
471	8.3841106	9.9019415	7.4734394
472	8.2216996	11.018523	7.7263243
473	8.4605339	7.7887747	7.5859645
474	10.420683	-1.731812	13.681597
475	-0.677371	1.5833266	4.3097423
476	5.3064049	6.5047221	7.3686239
477	7.3794582	-2.478652	7.8658504
478	-0.511304	8.7509915	7.9391442
479	7.6742154	2.0356592	10.831983
480	4.8822842	9.4845269	9.9783797
481	11.242144	10.501544	0.60961
482	9.8163666	0.6771168	5.813313
483	5.0003418	8.4175192	4.9397456
484	6.5670159	9.7520768	12.93503
485	10.397264	7.9152875	7.6609006
486	8.5509365	13.601394	5.1464478
487	11.737755	8.3972711	3.2881502
488	8.1314709	5.7808642	3.0417349
489	8.159862	-1.503817	3.3895702

490	1.0803154	1.4545529	8.8661986
491	1.9332972	6.6280893	7.0469318
492	7.5440306	6.9470313	10.962857
493	6.3272988	9.4759514	9.4649141
494	7.2950577	10.365826	10.660749
495	9.1897329	8.3484547	8.7882889
496	7.9185572	2.0206193	12.498553
497	3.1883983	12.435564	6.6389965
498	13.535031	12.629787	12.543511
499	11.836176	7.126167	9.9481882
500	9.8528544	7.0917733	2.0881274

Data simulated for Constant Elasticity of Substitution (CES)

($y = \gamma \left[\delta x_1^{-\tau} + (1-\delta) x_2^{-\tau} \right]^{-\frac{v}{\tau}} e^u$) **when N=50**

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	1.8578268	1.8351303	0.0975519
2	0.6384163	0.4511035	0.2487886
3	1.6148319	1.6162083	0.9379183
4	1.3137527	0.1819287	0.2618044
5	0.3054215	0.6623357	0.2678225
6	0.1761717	0.6815158	0.6205443
7	0.2963985	0.4248059	0.4536446
8	1.9696845	2.3237736	0.4775995
9	0.1529849	0.1899353	0.3486382
10	0.2063361	0.6805549	0.087247
11	0.5992771	1.558382	0.8863318
12	0.119956	0.2802609	0.3034601
13	0.1887658	0.9760454	0.8362578
14	1.5142838	1.0775896	0.6325666
15	0.9217854	0.3734826	0.1582934

16	0.7523281	1.686707	0.2291061
17	0.0315064	0.8887415	0.5165112
18	0.0320692	0.4494903	0.3168721
19	0.9227901	1.5753375	0.6737598
20	1.1115328	3.1265275	0.3136815
21	0.1165841	0.6256029	0.4005628
22	1.2004795	0.4172765	0.180514
23	1.7420059	0.1884906	0.5182508
24	0.6882129	0.3046012	0.7668122
25	0.7458361	2.6489283	0.8719089
26	0.0456423	1.3588565	0.1141403
27	0.0881328	0.9658883	0.8991795
28	0.4317612	3.8150151	0.4924753
29	0.754848	0.3914089	0.1436535
30	0.2762669	0.3007724	0.2529299
31	0.3147502	1.7301574	0.6347013
32	0.0732009	1.3339346	0.5276591
33	0.2686886	0.7813946	0.7044034
34	0.2729934	1.2843443	0.2592125
35	0.6502131	0.4212427	0.8819048

36	2.5607867	2.1869176	0.1216646
37	0.5960518	1.5219993	0.8401632
38	0.0941311	0.4351511	0.9423626
39	0.539863	10.295627	0.1046994
40	1.6724144	2.1010266	0.3382674
41	1.9227307	1.3220107	0.5750936
42	4.3576742	0.868892	0.5304978
43	0.9440014	3.5323797	0.6980249
44	0.0710798	1.2032884	0.4244927
45	0.4403758	2.2592376	0.6530688
46	0.0738823	0.5593955	0.7796689
47	0.8183749	0.2455327	0.5207123
48	0.058109	2.6487838	0.1193888
49	6.1660127	0.3809708	0.2485
50	0.4809336	1.3801024	0.6142457

Data simulated for Constant Elasticity of Substitution (CES)

($y = \gamma\left[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}} e^u$) when N=100

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	1.8230274	1.8351303	1.3372611
2	0.9122808	0.4511035	0.1966226
3	0.8258382	1.6162083	0.8971762
4	1.380858	0.1819287	0.3256733
5	0.2492515	0.6623357	0.0787894
6	0.2015545	0.6815158	0.0262731
7	0.2147684	0.4248059	0.3200009
8	1.5194332	2.3237736	0.080442
9	0.2036017	0.1899353	0.3618925
10	0.1431625	0.6805549	0.2295922
11	0.7066645	1.558382	0.0338827
12	0.0646871	0.2802609	0.0795211
13	0.3920507	0.9760454	0.0446885
14	1.2149641	1.0775896	0.0461808
15	0.7145132	0.3734826	0.6049862

16	0.3369369	1.686707	0.432419
17	0.0450519	0.8887415	0.1267962
18	0.0860511	0.4494903	0.5794678
19	1.2748886	1.5753375	5.0289599
20	1.0812916	3.1265275	0.0461542
21	0.1098898	0.6256029	0.874118
22	1.6189512	0.4172765	0.0653006
23	1.8637721	0.1884906	0.1509432
24	0.5613639	0.3046012	0.2076022
25	0.5650172	2.6489283	0.776007
26	0.0070524	1.3588565	0.4582658
27	0.101507	0.9658883	0.8672822
28	0.4431904	3.8150151	0.3175254
29	0.5861532	0.3914089	0.2246598
30	0.2185739	0.3007724	0.2060767
31	0.2004862	1.7301574	0.1486238
32	0.2068881	1.3339346	0.3000495
33	0.3783518	0.7813946	0.6227316
34	0.0980421	1.2843443	0.1871194
35	0.2648998	0.4212427	0.0557421

36	2.9372104	2.1869176	0.1248705
37	0.6289528	1.5219993	2.4510424
38	0.1018354	0.4351511	0.1918302
39	0.6136639	10.295627	0.9713698
40	1.381688	2.1010266	2.086898
41	1.6128563	1.3220107	0.2572308
42	4.1981834	0.868892	0.4582703
43	1.0946264	3.5323797	0.0425391
44	0.0502938	1.2032884	0.9509687
45	0.5992584	2.2592376	0.3715603
46	0.0643645	0.5593955	1.6812626
47	0.6651329	0.2455327	11.161777
48	0.0834404	2.6487838	0.1399456
49	3.6712619	0.3809708	0.2816468
50	0.356957	1.3801024	0.1572806
51	2.2485407	3.5468877	0.053635
52	0.5440185	0.1417793	0.0948895
53	0.3442773	0.204353	2.0106023
54	1.0352095	1.4035777	0.1872737
55	0.7104141	5.9844973	0.2452869

56	0.3058153	4.7923521	5.0928479
57	0.0771971	1.1503795	0.0607335
58	1.2816581	0.1640968	1.2574975
59	0.0678634	0.2687846	0.0687016
60	0.1649021	0.5312526	0.9062442
61	0.5836681	0.1716904	0.6395717
62	0.3379669	0.3055493	0.7996218
63	0.9932436	0.9169924	0.4130148
64	0.7461199	2.8911493	2.0306046
65	2.6384677	0.2878981	0.8000371
66	0.9955305	1.5443445	0.1142065
67	0.0998413	0.4018333	0.8453651
68	0.1392547	1.2088867	0.2709283
69	2.1799495	1.3807162	0.0665915
70	2.2805841	3.6395406	0.5105184
71	0.2623635	0.8224899	0.1647977
72	0.0776667	0.5363855	0.1175582
73	0.4865039	0.2682804	1.7013764
74	0.1582078	0.859347	0.4015315
75	0.7516003	1.9074863	0.2048022

76	1.0147364	1.790352	0.4536176
77	0.1335373	0.333152	0.4227522
78	4.5208735	1.8875058	0.0689732
79	1.4824723	1.8288528	0.1426361
80	0.7719143	1.9994788	0.8397301
81	0.5719339	4.0856678	0.1671125
82	0.0100552	2.4405481	1.2934966
83	1.6544831	1.058402	0.1722953
84	0.3119784	2.9728555	0.0520737
85	1.5416973	7.523527	0.2397796
86	0.3648878	0.4623293	0.013414
87	3.4180541	1.9394577	0.4360453
88	1.0630511	0.5882289	0.0272721
89	3.2893452	0.1199644	0.0407472
90	0.0223519	1.0230376	0.3733734
91	0.3826802	5.7845902	0.3246082
92	0.2661731	2.6878702	0.2731261
93	0.4189739	1.0267524	0.4864102
94	0.911327	4.3087578	1.7988995
95	2.3570589	1.2492891	2.2815196

96	0.1123079	2.4355698	0.6988119
97	0.3658163	2.9091854	0.4653489
98	0.0297242	0.1736562	0.1249678
99	0.4572717	0.8252114	0.4872869
100	0.3034661	0.7868619	0.409406

Data simulated for Constant Elasticity of Substitution (CES)

($y = \gamma\left[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}} e^{u}$) when N=150

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	1.6905517	1.8351303	0.3435764
2	0.6379688	0.4511035	0.212438
3	2.3455768	1.6162083	0.903156
4	1.0820229	0.1819287	0.3263268
5	0.3208792	0.6623357	0.0732998
6	0.3296736	0.6815158	0.0275116
7	0.3614611	0.4248059	0.1143376
8	2.1253606	2.3237736	0.1862457
9	0.044594	0.1899353	0.257378
10	0.1934741	0.6805549	0.2080824
11	1.0886425	1.558382	0.0802856
12	0.1763655	0.2802609	0.0857829
13	0.2452999	0.9760454	0.0463635
14	1.5293767	1.0775896	0.0960307
15	0.6328476	0.3734826	0.9037813

16	0.2243276	1.686707	0.4080616
17	0.0467608	0.8887415	0.1757157
18	0.1434367	0.4494903	0.1774754
19	1.4066849	1.5753375	3.2518514
20	1.5179576	3.1265275	0.0599732
21	0.0786641	0.6256029	0.2041313
22	0.9775831	0.4172765	0.1593281
23	1.3423072	0.1884906	0.2609398
24	0.4230745	0.3046012	0.1954524
25	0.4972103	2.6489283	1.6928753
26	0.0526886	1.3588565	0.332165
27	0.0768962	0.9658883	1.1154046
28	0.4615926	3.8150151	0.196277
29	1.0331044	0.3914089	0.2165981
30	0.1152144	0.3007724	0.2129742
31	0.0871884	1.7301574	0.1098453
32	0.0691068	1.3339346	0.2627842
33	0.4990499	0.7813946	0.2351442
34	0.122979	1.2843443	0.1481752
35	0.4303316	0.4212427	0.0834731

36	1.9392896	2.1869176	0.0023627
37	0.4390463	1.5219993	1.9501277
38	0.0569409	0.4351511	0.2699555
39	0.8503202	10.295627	1.0188503
40	0.750743	2.1010266	0.6653725
41	2.0400455	1.3220107	0.3440031
42	3.0153974	0.868892	0.5428317
43	0.9602745	3.5323797	0.0452109
44	0.0925956	1.2032884	1.0255376
45	0.4606778	2.2592376	0.4801442
46	0.0249136	0.5593955	1.5927319
47	0.4552064	0.2455327	11.242389
48	0.0974764	2.6487838	0.2505684
49	4.6397982	0.3809708	0.3083631
50	0.4145755	1.3801024	0.1279795
51	1.0454116	3.5468877	0.041508
52	0.2884317	0.1417793	0.1201341
53	0.1143217	0.204353	1.4014763
54	0.6977829	1.4035777	0.2557674
55	1.1107421	5.9844973	0.2722546

56	0.2461022	4.7923521	4.3575845
57	0.0961429	1.1503795	0.1282733
58	1.0706838	0.1640968	1.5439663
59	0.1872123	0.2687846	0.0897443
60	0.1231117	0.5312526	0.9404826
61	0.4888201	0.1716904	0.1785326
62	0.1795742	0.3055493	0.8426112
63	0.5204539	0.9169924	0.2196811
64	0.654085	2.8911493	2.0328056
65	3.7993788	0.2878981	0.5555248
66	1.4530959	1.5443445	0.1072694
67	0.0347431	0.4018333	0.1413394
68	0.0602073	1.2088867	1.0297139
69	2.791879	1.3807162	0.0747436
70	1.3572266	3.6395406	0.0039813
71	0.1995876	0.8224899	0.1258696
72	0.2400486	0.5363855	0.1448575
73	0.4438753	0.2682804	2.1769066
74	0.1793247	0.859347	0.4388825
75	0.8430739	1.9074863	0.1128953

76	0.5748533	1.790352	0.5198707
77	0.172192	0.333152	0.2107948
78	3.7142237	1.8875058	0.0644079
79	1.3607605	1.8288528	0.1750299
80	1.3418221	1.9994788	0.4297725
81	0.5466858	4.0856678	0.4113116
82	0.0697287	2.4405481	1.0262436
83	0.4407521	1.058402	0.420522
84	0.363618	2.9728555	0.0868497
85	1.4483818	7.523527	0.0924765
86	0.5467679	0.4623293	0.1120181
87	4.1364766	1.9394577	0.6127378
88	1.3694635	0.5882289	0.0224145
89	2.3286519	0.1199644	0.0437251
90	0.0529637	1.0230376	0.8788133
91	0.2694882	5.7845902	0.3121405
92	0.3308687	2.6878702	0.2928189
93	0.4746667	1.0267524	0.5965515
94	0.5422837	4.3087578	1.4239126
95	1.9605679	1.2492891	2.2461241

96	0.140085	2.4355698	0.2869866
97	0.5092266	2.9091854	0.4168278
98	0.1261696	0.1736562	0.1309164
99	0.3597844	0.8252114	0.4128985
100	0.2948011	0.7868619	0.5319433
101	0.4638808	3.1366245	2.4084351
102	0.4213873	0.6551667	0.0630975
103	0.0899569	0.4018118	0.1970879
105	0.4582818	0.5984612	0.5314822
106	0.1471105	1.0384206	0.5142509
107	0.8630995	1.8797422	0.3069339
108	0.0503518	0.8519268	0.2906454
109	0.9520819	2.7228023	0.3856061
110	2.2946072	0.8945413	1.1451533
111	0.9505854	0.9560105	0.1076309
112	0.5667023	0.5441512	0.0840415
113	0.0392896	1.2306955	0.3409563
114	0.7046583	0.1146396	0.2342284
115	0.2280697	0.6123721	1.051179
116	0.4136579	0.1926721	0.3427384

117	0.4838516	5.0372021	0.2013548
118	0.1073836	0.1943837	0.6883287
119	0.2894555	3.1451028	0.211516
120	0.297549	0.5184362	1.7050497
121	0.8743596	0.3789713	0.2710955
122	0.4591634	0.3557629	2.1687193
123	0.2966191	0.3059371	0.0406173
124	0.7448531	1.6993147	1.2894951
125	2.4073137	4.7828548	1.8871901
126	0.1770983	0.3177914	0.0819778
127	0.2920828	2.2670921	2.7546453
128	0.0917869	4.2205951	0.1834638
129	0.3235814	1.6458238	0.19444
130	0.0436292	1.559527	0.4090225
131	0.0719311	1.3885458	1.3440646
132	1.7626471	0.2147753	0.022029
133	1.6073486	0.2592646	0.1064053
134	0.8275247	0.5434325	1.0108494
135	0.123566	7.6414235	0.0205829
136	0.1120659	0.1623586	0.1065077

137	1.7067727	0.6841532	0.5130955
138	0.4963832	0.2983134	4.2534441
139	0.8760673	0.9257506	0.3133488
140	0.6317923	3.3527087	0.1689233
141	0.1635544	4.2055448	3.5446839
142	2.5754294	2.1592402	3.7930486
143	2.210534	0.3031012	0.2007038
144	0.8450053	1.2469155	0.477955
145	0.280216	6.230548	1.3777106
146	1.1365023	1.1892795	0.4275293
147	1.9225142	0.2071848	0.3732145
148	0.0497772	0.558383	1.5596886
149	0.3763821	1.57473	0.0693152
150	0.1486867	0.4047106	0.3779859

Data simulated for Constant Elasticity of Substitution (CES)

($y = \gamma\left[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}} e^{u}$) when N=250

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	1.57388284	1.83513027	9.234647E-01
2	0.47912022	0.45110350	1.516607E-01
3	1.76205680	1.61620827	6.782376E-01
4	1.02350362	0.18192867	5.392703E-01
5	0.40710228	0.66233574	6.160397E-02
6	0.35254481	0.68151577	2.767673E-02
7	0.20373166	0.42480593	1.078520E-01
8	1.86505632	2.32377365	1.509016E-01
9	0.21412156	0.18993529	3.936340E-01
10	0.18494217	0.68055489	1.267050E-01
11	0.79019189	1.55838199	2.178277E-01
12	0.11128227	0.28026092	7.467343E-02
13	0.37171527	0.97604544	5.026861E-02
14	1.44438385	1.07758961	1.116800E-01
15	0.84313407	0.37348260	5.851765E-01

16	0.57836149	1.68670699	3.001319E-01
17	0.01943467	0.88874153	1.516355E-01
18	0.01512541	0.44949033	4.228881E-01
19	1.82356495	1.57533750	3.283009E+00
20	1.15845117	3.12652746	6.569406E-02
21	0.10769274	0.62560286	8.722062E-01
22	2.37899804	0.41727653	1.889173E-01
23	1.45910129	0.18849061	2.424047E-01
24	0.67075599	0.30460124	2.384309E-01
25	0.29041053	2.64892826	2.472487E+00
26	0.02267255	1.35885652	3.310930E-01
27	0.06596097	0.96588827	4.177168E-01
28	0.22869076	3.81501515	3.845353E-01
29	1.11260855	0.39140890	3.141549E-01
30	0.15784991	0.30077239	1.147324E-01
31	0.22466747	1.73015742	2.365505E-02
32	0.21842451	1.33393457	2.056667E-01
33	0.28694940	0.78139463	5.594832E-01
34	0.05014746	1.28434429	1.255514E-01
35	0.62263625	0.42124266	1.043963E-01

36	2.41585160	2.18691757	1.404639E-01
37	0.43477300	1.52199928	2.620512E+00
38	0.04158867	0.43515112	2.171815E-01
39	0.75289678	10.29562667	1.088346E+00
40	1.78996320	2.10102660	1.879055E+00
41	1.44257150	1.32201069	3.967352E-01
42	4.40683804	0.86889203	3.638741E-01
43	1.31707310	3.53237966	3.765241E-02
44	0.02591650	1.20328841	6.743860E-01
45	0.37030800	2.25923760	4.028591E-01
46	0.03682010	0.55939547	1.871855E+00
47	0.79371910	0.24553269	1.050785E+01
48	0.21268888	2.64878381	7.025462E-02
49	3.18061547	0.38097080	2.707338E-01
50	0.38908547	1.38010238	1.419186E-01
51	3.02078088	3.54688765	1.082427E-01
52	0.55937285	0.14177930	1.306029E-01
53	0.30586415	0.20435303	2.067640E+00
54	0.58725298	1.40357770	4.666328E-03
55	1.09727874	5.98449730	2.820269E-01

56	0.30606511	4.79235211	4.561599E+00
57	0.05890658	1.15037946	1.421493E-01
58	0.77040499	0.16409679	7.992138E-01
59	0.14363741	0.26878457	2.548696E-02
60	0.07582652	0.53125262	7.635442E-01
61	0.40898840	0.17169043	1.859403E+00
62	0.30421939	0.30554930	8.009616E-01
63	0.55757095	0.91699240	2.855191E-01
64	0.58844626	2.89114929	1.438310E+00
65	1.66845114	0.28789811	5.672257E-01
66	1.77912330	1.54434452	3.058506E-02
67	0.04278642	0.40183329	8.473313E-01
68	0.14905913	1.20888673	6.664191E-01
69	1.47207519	1.38071618	9.110565E-02
70	2.10814750	3.63954061	5.229297E-01
71	0.42816659	0.82248992	6.618875E-01
72	0.19056389	0.53638554	1.416463E-01
73	0.40497043	0.26828044	2.048922E+00
74	0.21347814	0.85934697	1.081953E-01
75	0.65622759	1.90748632	2.160693E-01

76	0.88314680	1.79035199	4.747221E-01
77	0.09113969	0.33315202	5.480481E-01
78	4.43802152	1.88750578	1.006016E-01
79	1.08614199	1.82885280	3.857183E-02
80	0.99434439	1.99947884	9.198528E-01
81	1.12239402	4.08566779	2.458467E-01
82	0.06866278	2.44054806	6.806855E-01
83	1.13307435	1.05840204	2.983768E-01
84	0.42673163	2.97285550	3.573706E-02
85	1.71333197	7.52352702	1.809411E-01
86	0.41216271	0.46232930	1.493593E-01
87	4.11608058	1.93945774	6.077539E-01
88	1.47600067	0.58822892	1.124070E-02
89	2.53023999	0.11996437	2.900046E-02
90	0.04632995	1.02303756	8.876223E-01
91	0.26202515	5.78459016	4.218082E-01
92	0.22982244	2.68787017	2.823887E-01
93	0.63449322	1.02675241	2.879342E-01
94	0.92582300	4.30875781	1.806569E+00
95	3.31339580	1.24928909	1.246852E-01

96	0.14692475	2.43556976	3.630735E-01
97	0.23284778	2.90918539	2.902296E-01
98	0.06522628	0.17365619	1.153470E-01
99	0.51420672	0.82521144	4.493370E-01
100	0.28463636	0.78686185	3.932445E-01
101	0.67097328	3.13662452	2.576572E+00
102	0.31971894	0.65516670	9.462360E-02
103	0.09181549	0.40181180	2.178182E-01
105	0.23513780	0.59846115	5.971434E-01
106	0.10332158	1.03842063	7.641641E-01
107	0.53532781	1.87974220	2.143925E-01
108	0.07847553	0.85192676	8.087207E-01
109	0.30987577	2.72280234	2.667719E-01
110	2.59000803	0.89454129	5.902868E-01
111	0.77752463	0.95601054	1.294521E-01
112	0.70939416	0.54415118	7.115083E-02
113	0.03028614	1.23069547	4.531112E-02
114	0.62969606	0.11463963	2.026158E-01
115	0.28883921	0.61237211	9.896284E-01
116	0.23938565	0.19267207	5.123103E-01

117	0.16187780	5.03720214	2.188283E-01
118	0.12539649	0.19438372	8.293613E-01
119	0.99415806	3.14510281	3.436063E-01
120	0.28970328	0.51843619	1.953638E+00
121	1.11070162	0.37897127	2.568593E-01
122	0.36409582	0.35576295	2.258138E+00
123	0.23725949	0.30593713	4.474523E-02
124	0.61376178	1.69931474	1.517466E+00
125	2.06848726	4.78285477	1.710563E+00
126	0.34832412	0.31779139	7.554415E-02
127	0.41342541	2.26709213	3.915840E+00
128	0.13239999	4.22059510	4.733010E-01
129	0.46745724	1.64582378	1.918332E-01
130	0.04353701	1.55952702	4.133345E-01
131	0.05884576	1.38854581	2.185792E+00
132	2.11310239	0.21477526	2.003341E-01
133	0.87262203	0.25926460	1.524692E-01
134	0.95825716	0.54343253	4.922416E-01
135	0.33088874	7.64142350	1.130012E-02
136	0.08986995	0.16235861	8.281094E-02

137	2.27609336	0.68415324	3.675929E-01
138	0.38437026	0.29831343	4.646564E+00
139	1.48989916	0.92575061	3.262252E-01
140	0.62112976	3.35270873	1.785049E-01
141	0.10024292	4.20554481	3.797290E+00
142	4.13757679	2.15924021	3.787745E+00
143	1.65334826	0.30310117	3.022431E-01
144	0.54640927	1.24691551	4.962521E-01
145	0.56097739	6.23054801	1.998298E+00
146	1.85126190	1.18927951	4.029126E-01
147	2.78190415	0.20718481	1.833747E-01
148	0.03184704	0.55838301	8.552294E-01
149	0.44238908	1.57473002	1.543597E-01
150	0.24235620	0.40471059	2.963812E-01
151	0.13186081	1.05451923	9.864248E-01
152	3.93874381	1.66835651	2.641409E-02
153	0.70803949	0.81724850	1.151126E+00
154	1.01412644	4.70370977	1.033219E+00
155	0.19152279	2.47846623	1.957876E-01
156	0.37912149	3.08634632	8.514356E-01

157	0.88289045	2.61958950	2.561119E-01
158	0.29369487	0.13983014	3.945468E-02
159	0.89822555	4.09014985	1.828500E+00
160	0.01344011	0.35668660	6.462833E-01
161	1.03363198	0.61301111	2.565815E-01
162	1.18320313	0.12451518	2.773610E-01
163	4.36939297	2.98818159	7.216057E-01
164	0.06658692	0.53309555	1.424192E+00
165	0.58161016	0.45383994	1.411686E+00
166	0.63046588	3.15914889	1.069128E+00
167	0.35948715	1.12812423	1.590701E+00
168	0.18424139	0.97199825	2.023941E+00
169	0.19042419	1.29292514	7.253448E-02
170	0.16474154	2.58913645	1.384392E+00
171	0.34933521	1.67820726	8.664013E-02
172	0.27613112	17.03012161	1.259598E-01
173	0.63444356	0.75913491	1.530673E-01
174	0.41695379	0.44607278	2.954772E-01
175	0.31994296	2.05229154	1.429666E-01
176	0.36795186	6.25620493	1.159870E-01

177	1.42322183	1.63016117	9.630127E-01
178	1.22771774	1.93651654	4.237161E-01
179	1.09222614	1.11783820	1.864087E-01
180	1.90861237	1.12885924	2.584217E-01
181	0.08428048	2.74628822	1.943648E-01
182	0.86539772	4.28839610	1.831171E-02
183	0.04503599	0.86433498	1.908384E+00
184	0.56616873	0.23789099	2.061629E+00
185	0.47203905	0.98441063	3.767191E-01
186	0.23665135	0.36510059	1.715037E+00
187	0.86483926	1.62060004	1.452763E-02
188	0.41655174	0.41350437	3.433820E+00
189	0.22488369	2.73321913	3.635810E-01
190	1.69544537	1.12934364	3.855595E-01
191	1.14323106	1.42093254	2.629969E-02
192	0.31985924	2.07528363	1.153897E-01
193	0.27114297	0.19721541	1.797435E+00
194	0.24554761	0.81819813	5.728165E+00
195	0.18751003	0.57109599	8.527086E-01
196	0.09577702	0.23229695	5.882134E-01

197	0.15667157	0.47056014	5.560903E-01
198	0.10729475	0.39920535	6.361927E-01
199	0.21896882	2.43888966	3.985606E-01
200	1.10795445	0.69640816	9.285661E-01
201	7.31711623	1.38979720	7.854365E-01
202	0.04014573	2.21799351	2.122834E-02
203	0.67179966	2.12621456	1.555600E-01
204	0.12818569	0.28798802	1.034684E-01
205	0.07536760	0.27951760	4.385479E-01
206	0.95427507	0.40040793	6.130691E-01
207	0.67994070	1.05931530	1.508168E+00
208	0.97648370	0.24123646	8.010482E-01
209	2.67488876	0.79875033	6.242631E-01
210	0.43496738	0.27425789	8.587115E-02
211	0.08812831	1.06527624	1.725417E-01
212	1.71114127	0.88725666	4.387500E-03
213	0.27579116	2.03435934	2.937086E-01
214	1.07879721	0.16319779	1.127087E-02
215	0.16225468	0.57356420	1.257537E-01
216	1.95316902	0.22872618	9.342819E-01

217	6.24474333	0.95133686	7.731302E-01
218	0.03352200	2.08952499	2.054167E-02
219	0.10767523	1.10866802	3.915618E-01
220	0.35777665	0.09152781	3.382914E-01
221	0.04164929	6.36163207	6.667428E-04
222	0.27671255	0.37176404	4.305876E-02
223	1.27889766	0.22312335	6.869342E-01
224	0.11705118	0.44975770	5.215552E-01
225	0.46258948	0.26757714	1.188645E-01
226	2.69349402	0.68745269	7.512838E-01
227	0.14283661	2.74183432	1.083355E+00
228	0.66457648	0.17545266	4.316763E-02
229	1.05752637	2.66112486	5.039859E-01
230	0.10994676	0.23063344	2.465751E-01
231	0.61218687	2.09431157	7.580496E-02
232	0.06382981	1.69496164	1.436805E+00
233	2.02051763	0.68747272	2.521003E-01
234	0.10804871	0.23504653	5.109320E-01
235	0.19495197	1.01582546	2.570420E-01
236	0.30832631	1.02154051	6.237173E-01

237	1.59134653	0.59803342	2.063706E-01
238	0.88067443	0.40843038	3.171635E-01
238	0.24001793	1.10386271	5.268540E-01
240	0.07627921	0.90267625	4.697014E-01
241	1.05667648	0.46463054	6.930101E-01
242	0.08243088	0.27480502	7.213207E-01
243	0.44416146	0.95567101	2.773213E-01
244	0.25035241	1.48449718	1.335882E+00
245	0.29035423	0.64603117	1.507582E+00
246	2.70271907	0.32645713	7.420208E-01
247	0.04915360	0.59847241	1.336489E-01
248	0.44236499	2.40973779	9.722763E-01
249	1.98002895	0.34140962	1.112620E+00
250	6.41790991	1.57111199	6.071161E-02

Data simulated for Constant Elasticity of Substitution (CES)

($y = \gamma\left[\delta x_1^{-\tau} + (1-\delta)x_2^{-\tau}\right]^{-\frac{v}{\tau}} e^u$) when N=500

	$\tau = -0.5$	$\tau = 0$	$\tau = 0.5$
S/NO	y_1	y_2	y_3
1	1.31023583	1.83513027	0.897839858
2	0.66440090	0.45110350	0.181487852
3	2.41839469	1.61620827	0.922754743
4	1.35238194	0.18192867	0.660841939
5	0.51323235	0.66233574	0.048898695
6	0.24376172	0.68151577	0.02633511
7	0.22421971	0.42480593	0.278673176
8	2.08769514	2.32377365	0.137086418
9	0.22051092	0.18993529	0.408516105
10	0.15143154	0.68055489	0.231407623
11	0.92883786	1.55838199	0.085260416
12	0.06601250	0.28026092	0.10214759
13	0.21855781	0.97604544	0.038689029
14	0.88462967	1.07758961	0.120155707
15	0.46365110	0.37348260	0.922606622

16	0.08282792	1.68670699	0.342480659
17	0.01769714	0.88874153	0.142684387
18	0.02812067	0.44949033	0.006980083
19	1.85451206	1.57533750	4.007236532
20	1.51993956	3.12652746	0.070070147
21	0.07977567	0.62560286	0.323707029
22	1.46971880	0.41727653	0.196487712
23	1.87500456	0.18849061	0.246762182
24	0.30461113	0.30460124	0.095261155
25	0.39123640	2.64892826	1.358001334
26	0.05031258	1.35885652	0.443509978
27	0.09301338	0.96588827	0.603331784
28	0.36152681	3.81501515	0.366642432
29	0.34151259	0.39140890	0.31232733
30	0.19637683	0.30077239	0.199199933
31	0.21277994	1.73015742	0.293037866
32	0.12139651	1.33393457	0.272173217
33	0.38279889	0.78139463	0.479549924
34	0.13075587	1.28434429	0.223687862
35	0.48985633	0.42124266	0.105036983

36	3.25033422	2.18691757	0.076293529
37	0.64491942	1.52199928	2.906875681
38	0.08837190	0.43515112	0.041516429
39	0.81222330	10.29562667	0.694154293
40	1.54616609	2.10102660	1.35766883
41	2.07258192	1.32201069	0.330829513
42	3.76319238	0.86889203	0.455540669
43	1.34039060	3.53237966	0.043864804
44	0.06226024	1.20328841	1.028008183
45	0.51168695	2.25923760	0.482198484
46	0.04323902	0.55939547	1.635296871
47	0.45075503	0.24553269	7.417929416
48	0.37131991	2.64878381	0.122321282
49	3.22560240	0.38097080	0.27185101
50	0.54031347	1.38010238	0.1382245
51	2.55366788	3.54688765	0.116994786
52	0.60900398	0.14177930	0.124615114
53	0.31027258	0.20435303	1.717077093
54	1.16866517	1.40357770	0.160760244
55	0.74155643	5.98449730	0.059991709

56	0.18112923	4.79235211	3.614096737
57	0.04362629	1.15037946	0.059116256
58	1.23639168	0.16409679	0.857185472
59	0.17031217	0.26878457	0.074555124
60	0.13104571	0.53125262	0.920588362
61	0.44889021	0.17169043	1.813604157
62	0.26328813	0.30554930	0.8424736
63	1.52450194	0.91699240	0.241406153
64	0.74096243	2.89114929	2.537387118
65	3.87455227	0.28789811	0.711443557
66	1.76428392	1.54434452	0.116091433
67	0.07416061	0.40183329	0.83175197
68	0.13135851	1.20888673	0.892132413
69	3.20218661	1.38071618	0.09933307
70	1.87143470	3.63954061	0.263964091
71	0.33772406	0.82248992	0.499975982
72	0.09663565	0.53638554	0.036709757
73	0.61930070	0.26828044	2.178724527
74	0.16025359	0.85934697	0.252828464
75	0.83194014	1.90748632	0.313950361

76	0.94190305	1.79035199	0.408955725
77	0.13786683	0.33315202	0.251118628
78	2.85205570	1.88750578	0.065765714
79	0.99486324	1.82885280	0.190983116
80	1.56551021	1.99947884	0.925596536
81	1.07691867	4.08566779	0.162868062
82	0.08069494	2.44054806	0.095094875
83	0.65835385	1.05840204	0.531821801
84	0.39309250	2.97285550	0.075120318
85	2.03943809	7.52352702	0.317410026
86	0.63015178	0.46232930	0.206063891
87	1.89742383	1.93945774	0.518564327
88	0.95075581	0.58822892	0.022316379
89	2.58119276	0.11996437	0.044581446
90	0.05478887	1.02303756	0.707109484
91	0.21046318	5.78459016	0.195632037
92	0.32442232	2.68787017	0.222746215
93	0.53432381	1.02675241	0.433973924
94	0.40751384	4.30875781	1.513254405
95	0.88157228	1.24928909	1.692216145

96	0.18415672	2.43556976	0.610071825
97	0.46989431	2.90918539	0.279809867
98	0.10234195	0.17365619	0.055701281
99	0.32724018	0.82521144	0.34655359
100	0.35320414	0.78686185	0.506961054
101	0.51068584	3.13662452	3.960717825
102	0.59856672	0.65516670	0.131774197
103	0.13072227	0.40181180	0.268768399
105	0.43910744	0.59846115	0.593354637
106	0.15829619	1.03842063	0.780176482
107	0.70241802	1.87974220	0.110712168
108	0.15182835	0.85192676	0.129787923
109	0.59889407	2.72280234	0.499385885
110	1.78318822	0.89454129	1.154809817
111	1.07631798	0.95601054	0.241043821
112	0.40153115	0.54415118	0.051516127
113	0.02815598	1.23069547	0.446305852
114	0.64764268	0.11463963	0.107924963
115	0.26721838	0.61237211	1.085203196
116	0.18457873	0.19267207	0.118480199

117	0.40894361	5.03720214	0.210125904
118	0.17103617	0.19438372	0.755149135
119	0.53366290	3.14510281	0.217562678
120	0.17930622	0.51843619	0.672885914
121	0.80387224	0.37897127	0.011475211
122	0.43465325	0.35576295	0.136729255
123	0.22080784	0.30593713	0.043197502
124	0.62793110	1.69931474	1.521308529
125	0.79617895	4.78285477	1.925767157
126	0.44419591	0.31779139	0.087121741
127	0.79213801	2.26709213	3.403829278
128	0.13279965	4.22059510	0.438798598
129	0.21088539	1.64582378	0.013644003
130	0.04145692	1.55952702	0.415316533
131	0.07168220	1.38854581	2.517931014
132	1.40888386	0.21477526	0.00273826
133	0.79053386	0.25926460	0.152886324
134	0.73211808	0.54343253	1.163309298
135	0.17185339	7.64142350	0.041926694
136	0.01779343	0.16235861	0.093548746

137	0.84648014	0.68415324	0.466578872
138	0.60020551	0.29831343	3.44548592
139	0.84700359	0.92575061	0.32276842
140	0.69262795	3.35270873	0.09837888
141	0.11465457	4.20554481	0.017945857
142	2.21197931	2.15924021	3.887125441
143	2.15062069	0.30310117	0.239100819
144	0.46065926	1.24691551	0.534615106
145	0.18939230	6.23054801	1.510818525
146	1.28177588	1.18927951	0.038796291
147	2.26618620	0.20718481	0.242914168
148	0.03263280	0.55838301	1.631319583
149	0.29087262	1.57473002	0.184678632
150	0.19188165	0.40471059	0.430830865
151	0.13194155	1.05451923	1.407523794
152	2.47281109	1.66835651	0.033388513
153	1.25671613	0.81724850	1.175600625
154	1.86454968	4.70370977	0.963256259
155	0.18648460	2.47846623	0.225195054
156	0.40203034	3.08634632	0.421199536

157	1.07768431	2.61958950	0.319451171
158	0.31686057	0.13983014	0.054525055
159	1.68230551	4.09014985	1.984136246
160	0.00047856	0.35668660	0.898044463
161	0.53755077	0.61301111	0.854248052
162	0.79900359	0.12451518	0.246712162
163	3.43480606	2.98818159	0.504166866
164	0.06003685	0.53309555	1.368401089
165	0.40842342	0.45383994	0.200946597
166	0.82693208	3.15914889	1.215884467
167	0.59224622	1.12812423	1.173511058
168	0.19701671	0.97199825	1.921430151
169	0.16482923	1.29292514	0.086314164
170	0.09315410	2.58913645	0.586891828
171	0.35461533	1.67820726	0.115984942
172	0.26505199	17.03012161	0.125037023
173	0.78794708	0.75913491	0.121708958
174	0.32299723	0.44607278	0.131675004
175	0.75630485	2.05229154	1.02032642
176	0.37404651	6.25620493	0.127052399

177	1.34116535	1.63016117	0.768257351
178	2.00254617	1.93651654	0.980668452
179	0.77159445	1.11783820	0.34942106
180	1.63056190	1.12885924	0.269292804
181	0.10585110	2.74628822	0.008614073
182	0.77303699	4.28839610	0.04629995
183	0.03121388	0.86433498	1.77428432
184	0.57009585	0.23789099	1.679834172
185	0.42753621	0.98441063	0.369929818
186	0.29751355	0.36510059	0.461766989
187	0.31303198	1.62060004	0.013355976
188	0.40400903	0.41350437	2.300840098
189	0.08186654	2.73321913	0.62299901
190	1.05520169	1.12934364	0.470521578
191	0.74286882	1.42093254	0.03878651
192	0.22923693	2.07528363	0.241434356
193	0.10700162	0.19721541	1.388039098
194	0.24789712	0.81819813	0.267736444
195	0.17932864	0.57109599	0.814686448
196	0.24235372	0.23229695	0.044934092

197	0.14725233	0.47056014	0.970164122
198	0.12576210	0.39920535	0.649982486
199	0.37202443	2.43888966	0.53422574
200	0.73537734	0.69640816	1.909356457
201	6.01748866	1.38979720	0.539720107
202	0.02961957	2.21799351	0.059884643
203	0.38141300	2.12621456	0.090535429
204	0.24459443	0.28798802	0.095282516
205	0.09133692	0.27951760	0.530991739
206	1.49629421	0.40040793	0.479005628
207	0.97781919	1.05931530	0.397668248
208	0.75295829	0.24123646	0.881321318
209	2.44611297	0.79875033	0.816830451
210	0.54339083	0.27425789	0.139891616
211	0.14281051	1.06527624	0.303906073
212	1.06487855	0.88725666	0.004509072
213	0.15264394	2.03435934	0.102758722
214	1.18033403	0.16319779	0.03048856
215	0.14763919	0.57356420	0.273997839
216	2.09403564	0.22872618	1.55309325

217	9.57745590	0.95133686	0.82610136
218	0.01871774	2.08952499	0.022149758
219	0.08995686	1.10866802	0.410745154
220	0.41540014	0.09152781	0.539273262
221	0.02518734	6.36163207	0.00065461
222	0.40467498	0.37176404	0.065683293
223	0.97809531	0.22312335	0.879679188
224	0.17718544	0.44975770	0.513522686
225	0.24622601	0.26757714	0.05042962
226	1.76934989	0.68745269	1.042164513
227	0.05045471	2.74183432	1.319142302
228	1.06507442	0.17545266	0.048127414
229	0.62564463	2.66112486	0.544616766
230	0.15521399	0.23063344	0.25029058
231	0.51784268	2.09431157	0.068409313
232	0.04979541	1.69496164	1.50899051
233	0.84954189	0.68747272	0.358523024
234	0.03907693	0.23504653	0.79306084
235	0.17149481	1.01582546	0.436109326
236	0.24727987	1.02154051	0.527867513

237	0.87039243	0.59803342	0.108674325
238	0.63356625	0.40843038	2.025000286
238	0.25030953	1.10386271	1.493206416
240	0.07124957	0.90267625	0.517360819
241	1.08102549	0.46463054	0.186628787
242	0.11366815	0.27480502	0.590366145
243	0.70214906	0.95567101	0.841281153
244	0.19462643	1.48449718	1.51699293
245	0.30341009	0.64603117	2.298295465
246	1.80546867	0.32645713	0.563846157
247	0.07897385	0.59847241	0.127327968
248	0.27715765	2.40973779	0.684328844
249	1.54664335	0.34140962	0.806031775
250	3.60701156	1.57111199	0.19963294
251	0.97176254	2.59683692	0.079729204
252	0.10528723	3.07220474	0.196727114
253	0.27457067	6.59910009	0.344807463
254	0.35534392	1.21956492	0.293216235
255	0.84752920	0.33518183	0.310568105
256	0.21255843	4.92993628	0.585681068

257	0.86057365	2.40984003	0.372995431
258	1.74656717	0.47179999	0.247240446
259	0.13944147	1.40300391	0.174232492
260	0.44927360	6.63520124	0.146644169
261	1.74706967	0.56482170	0.254423258
262	0.23955763	1.96067234	1.570222293
263	0.85629457	2.68618575	0.120132041
264	0.78634930	1.54758391	0.548820874
265	0.11253450	3.83776768	0.412273745
266	0.18499983	0.66692447	0.20731866
267	1.12544076	0.49037665	0.254956272
268	0.43130657	0.59819781	0.591982738
269	0.18624099	2.57782076	0.149907603
270	1.90125860	0.16410962	0.444761201
271	0.20535999	1.78996861	0.961716836
272	0.19867348	1.55128823	0.150153235
273	0.25157710	0.75263413	3.260151231
274	0.95158029	0.69386094	0.341167222
275	0.30641354	0.34664135	2.677087438
276	0.30516122	2.45396568	0.063511941

277	0.38818921	0.82027740	0.595650911
278	0.22214933	0.19046257	0.366056746
279	0.22765289	0.15136198	0.16823646
280	0.74556582	1.15439861	0.747598646
281	0.90779349	4.18751020	0.11947864
282	0.35757098	2.36095348	0.002224461
283	0.20355420	0.30385216	0.437637487
284	0.98324198	1.51195690	0.221801619
285	0.16877153	1.31162604	0.49137328
286	0.65859012	0.13660847	0.33245213
287	1.09662446	0.20774014	1.442783667
288	0.54675571	0.24914136	4.00734891
289	0.04374754	4.51541703	0.960045456
290	0.76841615	3.04523823	1.123683126
291	0.88660279	1.65879891	1.449708595
292	0.07721682	0.44670541	0.226720099
293	0.70544406	3.70323148	6.836891388
294	0.58069128	0.52903986	0.283820943
295	0.86314783	0.32533214	0.451998466
296	0.51890171	1.99250283	0.167313091

297	0.58998375	0.99701643	1.14283099
299	3.33142369	6.24784049	0.141576695
300	1.20430045	2.00856217	0.45254145
301	1.08569800	1.31221829	1.752067517
302	1.32654468	10.38912569	0.045428899
303	0.27367748	0.32587098	0.430370136
304	0.14369209	0.48726634	0.096268434
305	1.31068275	0.40525094	0.365767675
306	0.00918309	1.52819967	0.001667149
307	0.16919029	0.59476092	0.27244028
308	0.18216295	0.24663836	0.501592468
309	0.53452838	2.79937180	1.587758529
310	0.23349925	2.21839758	0.370558283
311	0.28260205	0.99721964	0.321717074
312	0.37472961	1.04907136	0.276654209
313	0.12887991	1.05987695	0.176656327
314	3.08573306	2.08586219	5.016010221
315	0.02161575	1.36374140	0.132710963
316	0.13901462	1.06617562	0.841035102
317	0.10893696	1.10300865	0.157909462

318	0.58955827	0.89885599	0.053243013
319	0.09553340	2.82548766	2.471239623
320	0.34241787	0.43503367	0.321029643
321	0.19818960	0.92234355	0.302431316
322	3.53447438	1.65176425	0.044000995
323	0.57724749	1.72967689	0.353912369
324	0.57870934	2.65123314	0.160400424
325	0.33871542	0.54143174	1.873099001
326	1.95485149	0.24549651	0.104157583
327	0.02605724	0.70140404	0.266978771
328	0.55490752	1.52589832	0.081724062
329	0.89561721	0.70478847	0.132249464
330	0.89657432	3.80064683	0.183552662
331	0.28632838	0.27403839	0.006669027
332	0.91360167	1.40751554	0.279131936
333	0.59242111	0.47631271	0.137748746
334	0.40156514	0.40264970	0.224982852
335	0.42248512	0.19325553	0.582176453
336	0.56932687	0.56542882	0.174434319
337	0.34023249	4.17816706	0.151778682

338	0.54989998	0.19542999	0.00191215
339	0.42852783	0.89083274	0.260283455
340	1.32542443	4.81960901	0.3623419
341	0.09828439	0.25926016	0.129878315
342	0.14890401	0.29621231	0.657549491
343	0.21517857	0.56777262	0.3670403
344	0.44792342	1.57850818	0.557043737
345	1.91318414	0.66539878	0.107246626
346	0.49136780	1.77737555	0.640267292
347	0.21474352	2.21251946	0.011319538
348	0.33141807	0.32820660	0.063375334
349	0.40914394	5.82371829	1.178469838
350	1.17516933	0.28381033	0.82619131
351	1.64995610	0.85055410	1.182815088
352	2.48971768	1.38558360	0.245846807
353	9.03621796	3.79828431	0.029869299
354	0.26167533	0.94513068	0.285016286
355	0.34126050	0.84332345	0.798830724
356	0.06170160	0.26479142	3.19802787
357	0.25818622	0.39473593	0.894404212

358	0.82586270	9.04834147	0.179705745
359	0.17028331	2.28871163	0.454033707
360	0.13605424	0.14847523	0.16162541
361	0.31802186	0.50548431	0.197792103
362	0.76375066	7.73710310	0.894550107
363	1.01209298	0.61667414	2.589570704
364	0.27109805	3.36298044	0.03595957
365	0.22518697	9.01104214	0.325080385
366	0.13653674	2.34522137	1.00378985
367	0.27217091	1.23286113	0.125767974
368	0.05424623	2.82353228	0.285139398
369	0.16329331	3.33316414	0.313721293
370	0.13330512	1.69034230	0.179678605
371	0.50456528	2.58549529	0.728181267
372	1.13718485	0.68579960	0.607849201
373	0.15523913	0.21575964	5.096413961
374	0.20664016	2.78286742	0.540586093
375	0.52280086	6.13357725	0.162790368
376	1.27627087	15.44421020	0.840043024
377	0.62369588	0.84987974	0.315498025

378	0.79861675	0.56847264	0.278834089
379	0.20973540	0.06020366	0.450037057
380	4.34122778	0.75505265	0.331847258
381	0.78982795	4.08601831	0.192500261
382	0.10784826	0.41263735	0.301159779
383	0.34407466	3.54866436	0.7965799
384	0.42000887	1.19179713	0.622757793
385	0.37603629	0.26829133	0.445675109
386	0.27162189	4.29112871	0.157958195
387	0.19254176	6.19195642	1.326683965
388	0.48823364	0.39713408	0.75777176
389	0.02611314	2.92562251	0.105643089
390	0.06987419	0.94074961	0.207642332
391	0.44969000	2.66973077	0.747662899
392	0.60075253	1.21588939	0.329808602
393	0.23273315	2.26465814	7.778514973
394	0.19280216	0.71396448	0.779040676
395	0.24760980	1.13413514	0.193174101
396	5.99300615	0.30263331	0.345733577
397	0.09991067	0.21104123	0.318775728

398	1.88537740	1.12824998	0.556164483
399	0.15642479	0.67232382	0.575037344
400	0.14549787	0.36770580	0.669600246
401	0.53584853	1.20599874	1.312124573
402	0.21886968	0.60187508	2.980220832
403	0.63184577	0.74732237	0.770121458
404	0.02842339	1.68995491	0.012021442
405	0.02716738	3.44727248	0.911936579
406	0.96782194	0.79145874	0.110290914
407	0.98320073	0.12851294	0.517088183
408	0.08376161	5.66512142	1.478537021
409	0.21351212	0.74938149	0.412132391
410	0.87152294	1.91053470	1.492592251
411	0.57733305	2.23378261	0.119675424
412	0.39211634	2.79347097	0.058237516
413	0.24395502	2.15465844	3.635197748
414	0.30815820	1.99954125	0.896896315
415	1.04544449	0.13576431	0.123175552
416	0.51731922	3.61698259	0.069768677
417	0.31172480	5.16336253	0.530203693

418	0.05247353	2.13942790	0.932487906
419	9.73508196	0.58023120	0.994447098
420	1.30066391	1.11682190	1.408243202
421	1.08553949	1.13331994	0.355518821
422	0.31358866	3.14362203	0.076891131
423	0.42040587	1.93153956	0.290480873
424	0.07584407	0.28979047	1.165262678
425	1.29287429	3.15075598	0.160033485
426	0.57697241	0.36353309	0.04603112
427	0.11772265	0.31905305	5.331809104
428	0.61336005	1.78216039	0.99820578
429	0.39712836	1.91681366	0.450432677
430	0.25592198	0.65304285	0.059056074
431	5.09888618	2.76857894	0.116030473
432	0.78461302	0.69790731	0.199001132
433	0.56780976	0.77775902	0.022571879
434	0.84963276	1.10636844	0.563705318
435	0.21131740	5.35563472	0.740226149
436	0.27420183	1.12139760	0.185889592
437	0.14276108	1.38867199	0.093156624

438	0.17969234	0.56611969	0.468405863
439	0.50168250	0.33083003	0.261753933
440	0.68941542	1.01923359	0.215215492
441	0.13810589	3.03403006	0.29151078
442	0.20887297	6.89851445	0.428740181
443	0.26296315	0.34859071	0.035817189
445	0.55641283	1.07385139	0.694282727
446	8.89748169	0.12962506	0.169457225
447	0.22890220	1.43838971	0.374032813
448	0.17414176	0.46017908	3.163588846
449	2.90673438	1.52316861	0.138133688
450	0.25920218	0.65149469	1.080745284
451	0.47297721	0.88867878	0.279218667
452	3.18906780	0.28385492	0.336128618
453	2.56718954	0.40604416	0.675911759
454	0.33054260	7.04227133	0.004710845
455	0.21631004	1.62455751	0.045663728
456	1.84212248	1.47986634	0.172505229
457	0.01833051	0.88057781	0.211693862
458	0.03789979	0.78168238	0.682475123

459	0.21630776	2.31976378	0.806584913
460	0.81437124	3.92694092	0.30668026
461	0.14528704	1.51233633	0.291092328
462	0.43931314	0.42169408	0.360672767
463	0.77593369	1.97753900	1.347410434
464	3.82453698	3.73951710	0.906535164
465	1.98861586	0.57135186	0.244699341
466	0.32631637	0.17679264	0.200709005
467	1.30359123	0.52279400	0.050516974
468	0.18850638	0.26164968	0.070459349
469	0.11400699	0.94282367	0.079839076
470	0.01379535	0.51772371	0.095984328
471	6.07409672	4.14938127	1.064162546
472	0.12308232	0.69394268	0.563894884
473	0.52669569	1.54530047	1.120293727
474	0.21694499	0.29518059	0.32735723
475	0.53426852	0.70401008	0.020552617
476	0.82025649	0.51890314	0.24155013
477	2.02380448	1.57238776	2.341904953
478	0.15430252	2.42630605	0.116239509

479	0.26115560	0.34144338	0.877943406
480	0.22423829	0.29477961	0.27804542
481	0.20049621	3.71398081	0.606552794
482	0.22551287	0.33167872	0.530803421
483	1.45929652	3.95278910	0.104361391
484	0.06975176	0.22652804	0.422460176
485	0.74941313	1.68574776	0.15731098
486	0.55729247	0.59084364	1.129788044
487	0.56375736	4.45682110	0.607173241
488	1.20969358	1.19676004	0.346999385
489	0.62752189	1.52339134	3.87456085
490	0.22624496	1.05137041	0.006714015
491	0.43705469	1.41472621	0.291378639
492	0.24114417	1.01621406	0.886578827
493	8.11488721	0.18866993	1.914292758
494	2.26331767	0.70742105	0.562467685
495	0.49848505	1.06063541	0.191849989
496	2.37583495	0.44644107	0.948950391
497	0.18696992	0.89571792	0.876601724
498	1.93980465	0.62477158	0.139499484
499	0.11293445	0.75853447	0.193851784
500	0.18490648	0.74377931	1.283982925

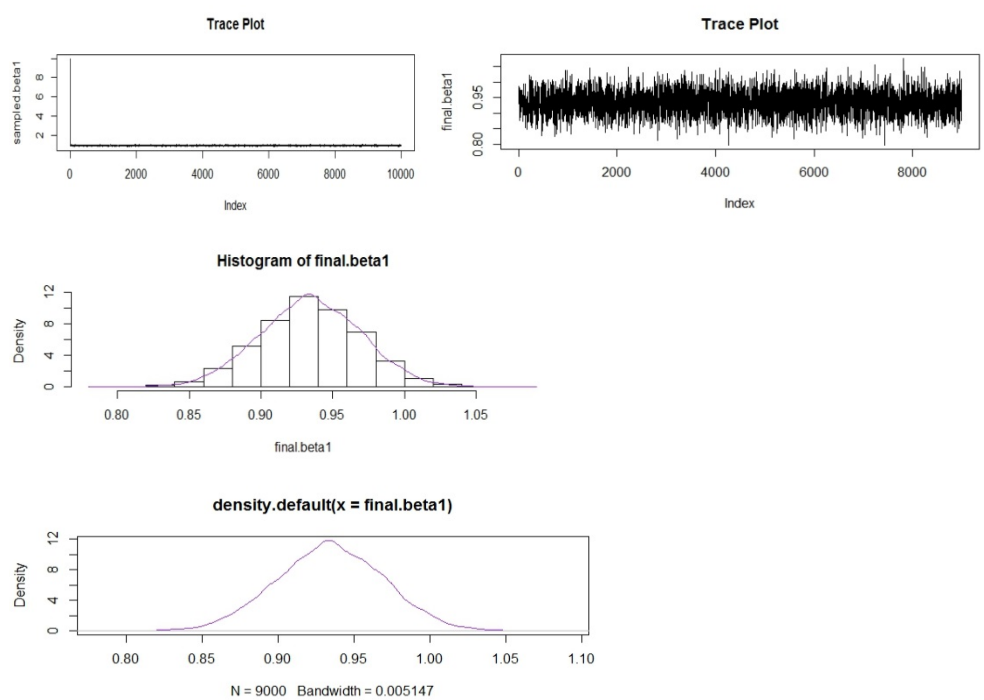
APPENDIX C

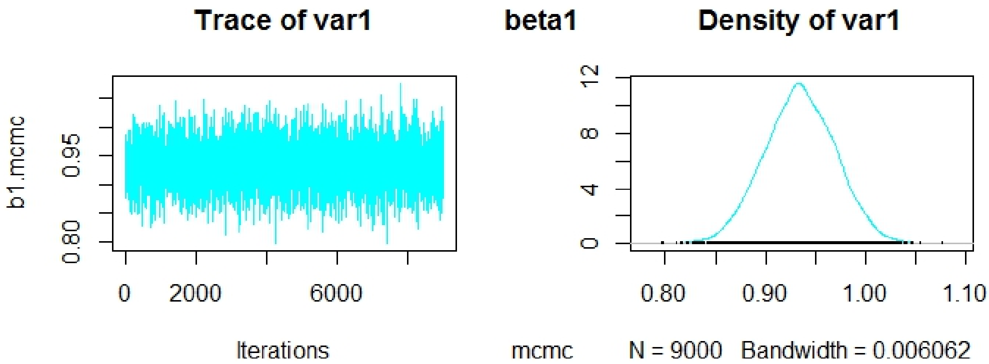
THE GRAPHICAL REPRESENTATION OF THE METROPOLIS-WITHIN-GIBBS TECHNIQUE

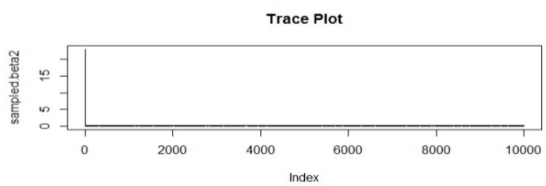
The illustration below shows the graphical representation of the Metropolis-within-Gibbs sampling before and after burn-in, alongside with the histogram of each posterior estimate for all the sample sizes considered, the figures labeled sampled beta $_j$represent the before burn-in while the parts labeled final beta $_j$ represent the after burn-in process.

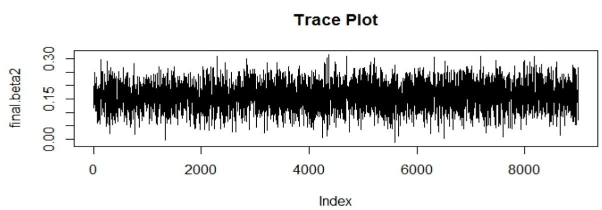
For Cobb-Douglas production with additive error term, when the production function assumed a constant returns to scale that is ($\beta_1(0.85)$, $\beta_2(0.15)$)

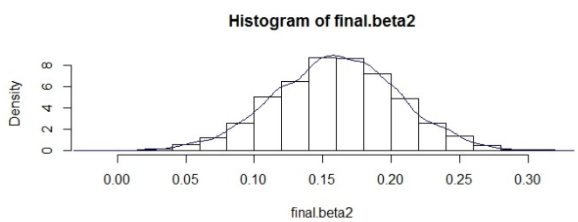
Figure 1.0: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=50

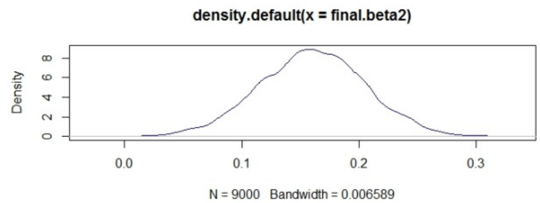



Trace of var1
beta1
Density of var1
b1.mcmc
0.80
0.95
0
2000
6000
Iterations
12
8
4
0
0.80
0.90
1.00
1.10
mcmc
N = 9000 Bandwidth = 0.006062


Trace Plot
sampled.beta2
0
5
15
0
2000
4000
6000
8000
10000
Index


Trace Plot
final.beta2
0.00
0.15
0.30
0
2000
4000
6000
8000
Index


Histogram of final.beta2
Density
0
2
4
6
8
0.00
0.05
0.10
0.15
0.20
0.25
0.30
final.beta2


density.default(x = final.beta2)
Density
N = 9000 Bandwidth = 0.006589

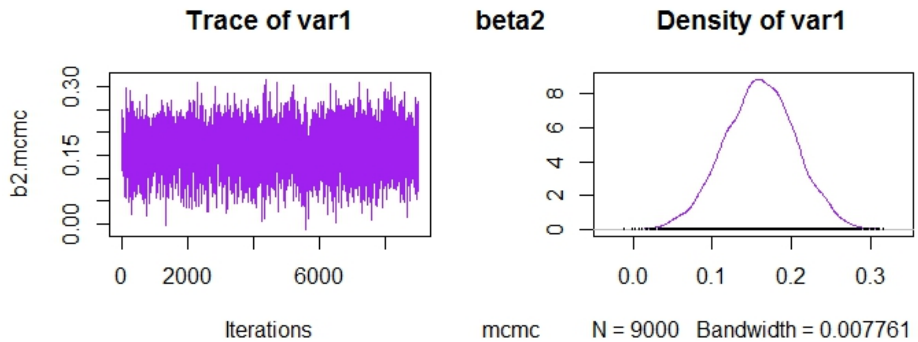
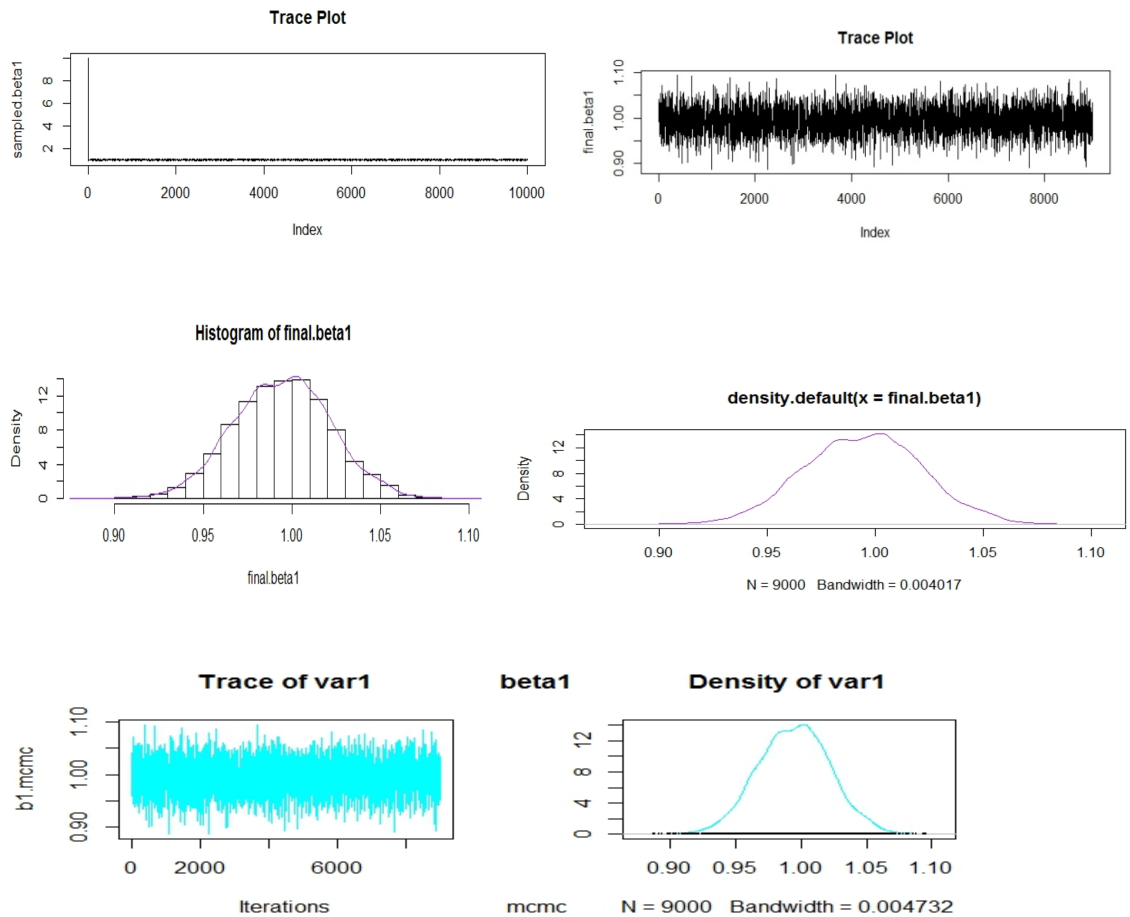
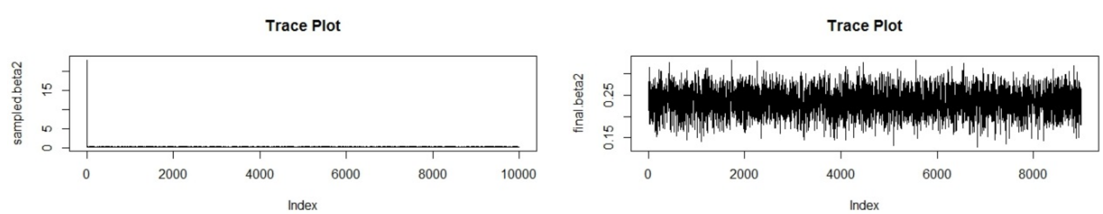
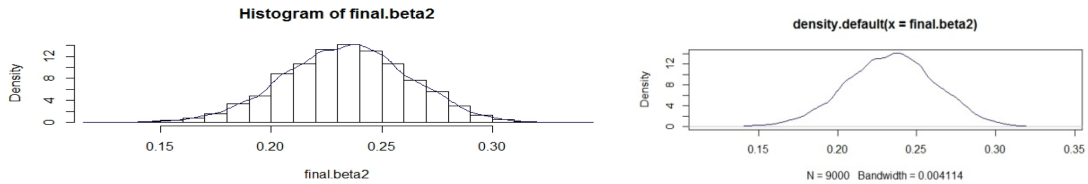

Trace of var1
beta2
Density of var1
b2.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.007761

Figure 1.2: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=100






Histogram of final.beta2
Density
final.beta2
density.default(x = final.beta2)
N = 9000 Bandwidth = 0.004114

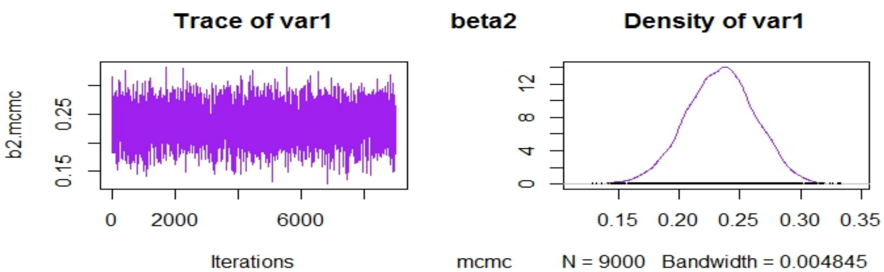
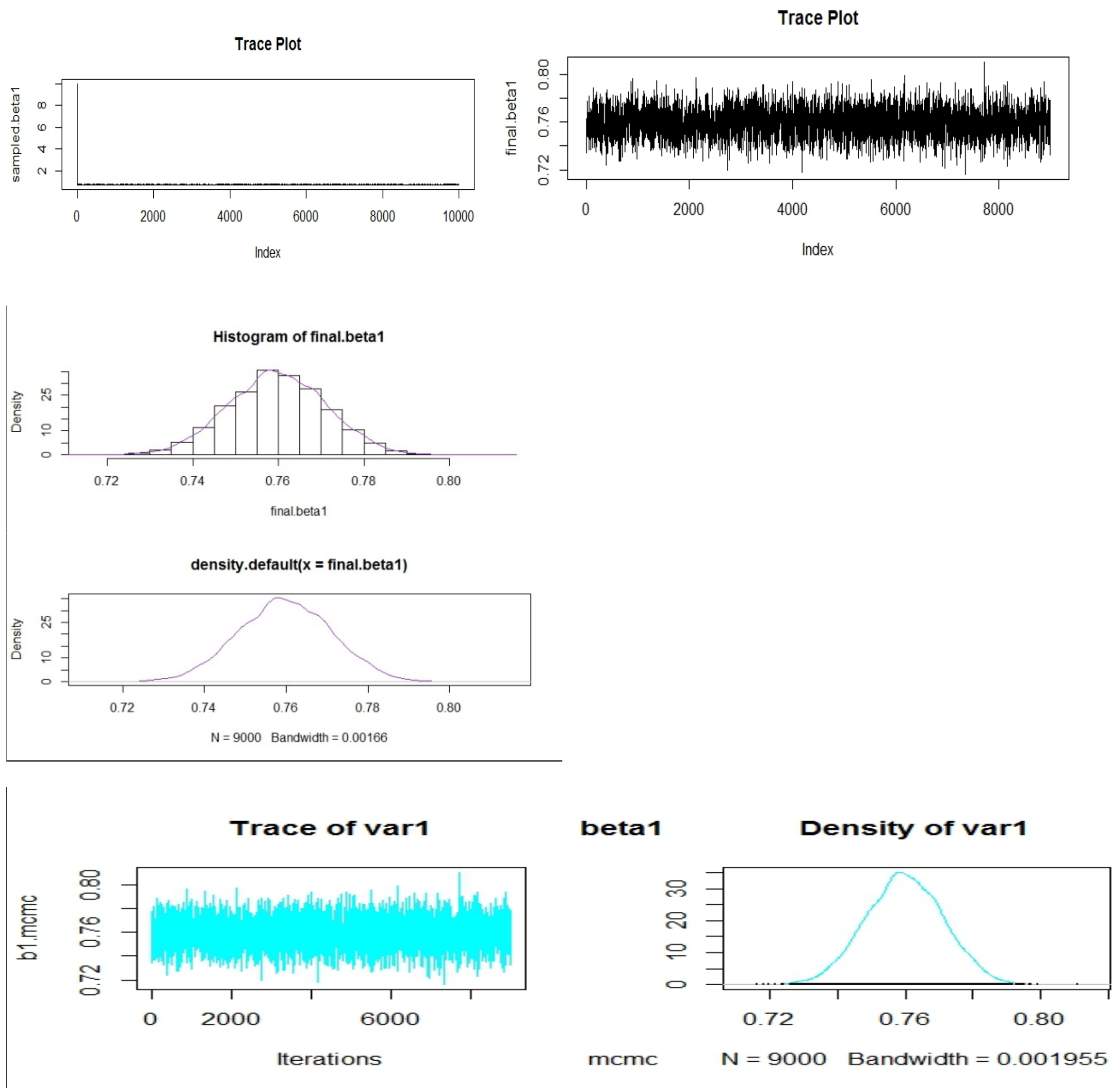
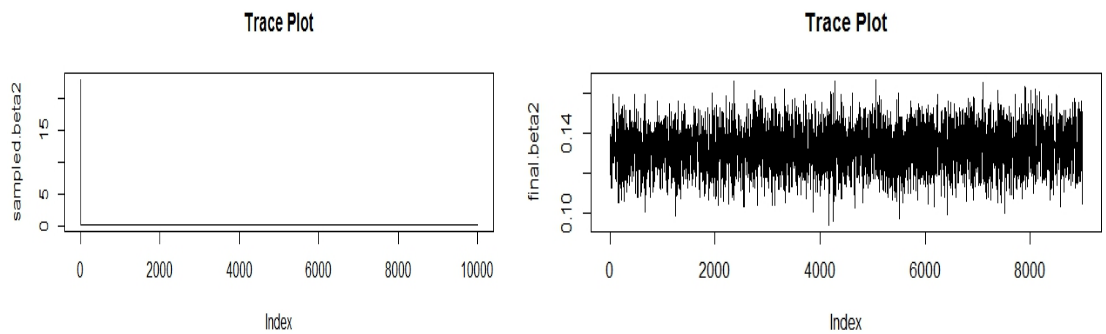
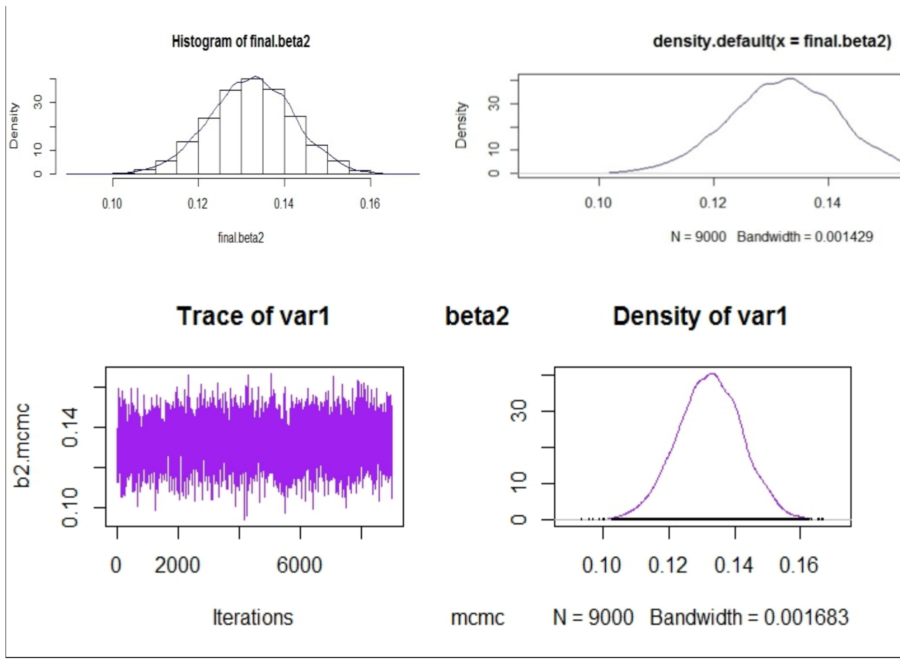

Trace of var1
beta2
Density of var1
b2.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.004845

Figure 1.3: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=500

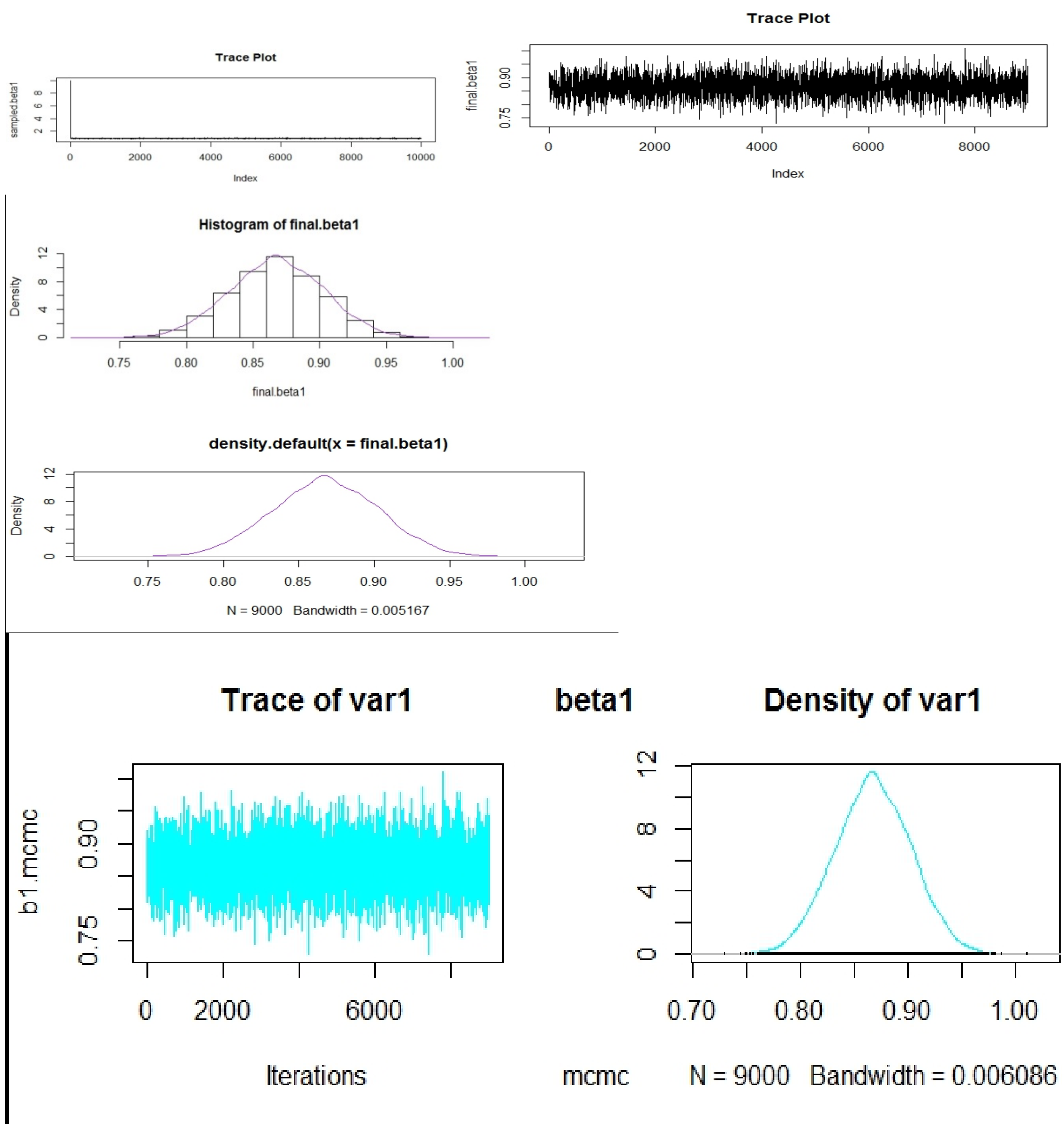


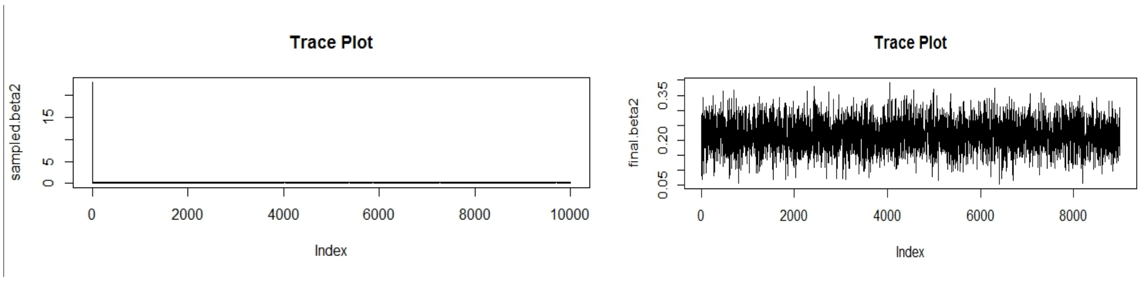

Trace Plot
sampled.beta2
Index
Trace Plot
final.beta2
Index


Histogram of final.beta2
Density
final.beta2
density.default(x = final.beta2)
Density
N = 9000 Bandwidth = 0.001429
Trace of var1
beta2
Density of var1
b2.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.001683

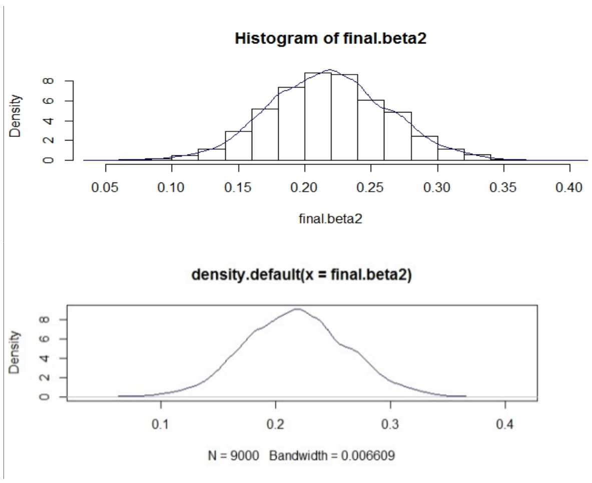
For Cobb-Douglas production with additive error term, when the production function assumed a increase returns to scale that is ($\beta_1(0.90)$, $\beta_2(0.20)$)

Figure 2.0: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=50

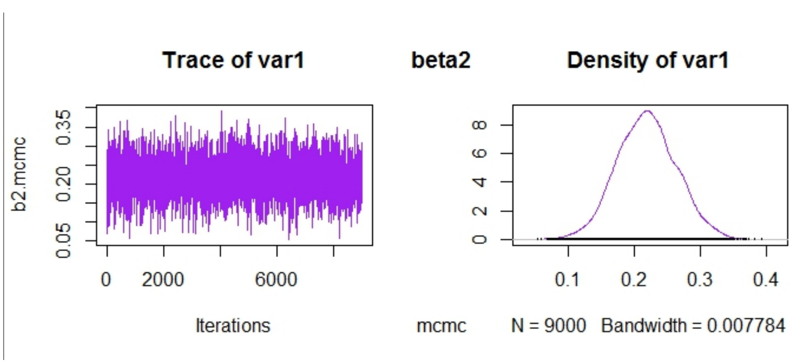




Trace Plot
sampled.beta2
0 5 15
0 2000 4000 6000 8000 10000
Index
Trace Plot
final.beta2
0.05 0.20 0.35
0 2000 4000 6000 8000
Index

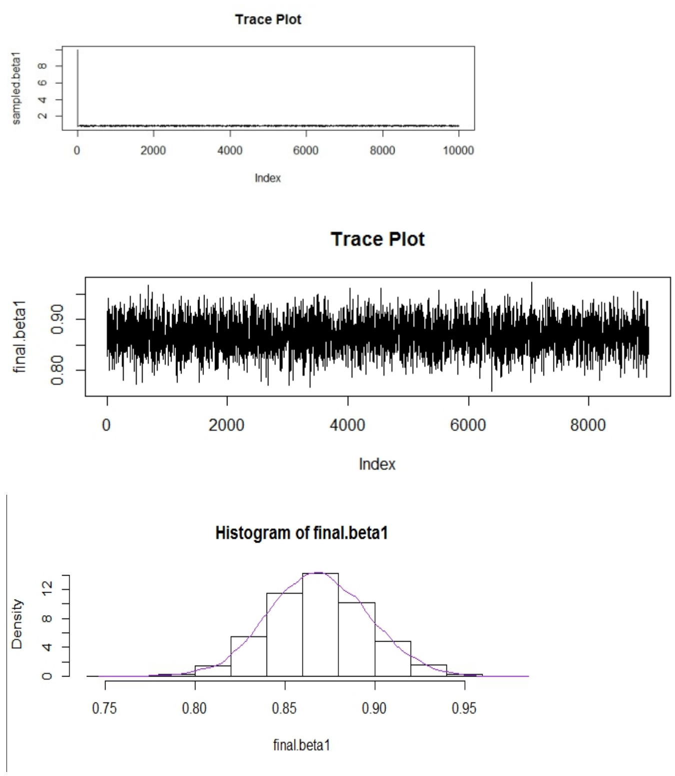


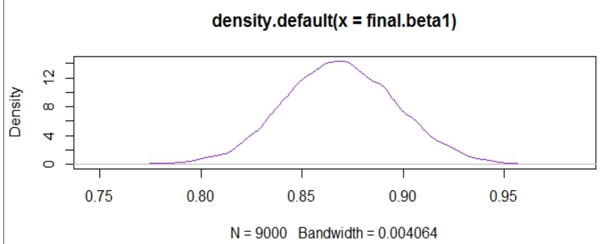
Histogram of final.beta2
Density
0 2 4 6 8
0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40
final.beta2
density.default(x = final.beta2)
Density
0 2 4 6 8
0.1 0.2 0.3 0.4
N = 9000 Bandwidth = 0.006609

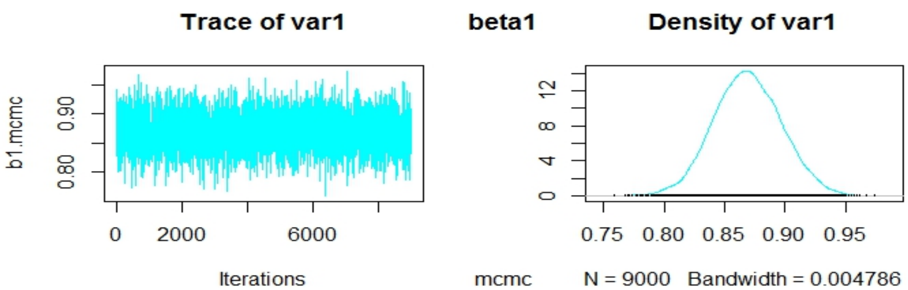


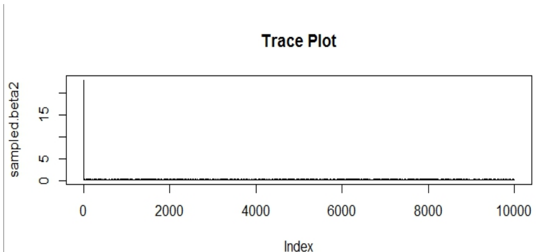
Trace of var1
beta2
Density of var1
b2.mcmc
0.05 0.20 0.35
0 2000 6000
Iterations
mcmc
0 2 4 6 8
0.1 0.2 0.3 0.4
N = 9000 Bandwidth = 0.007784

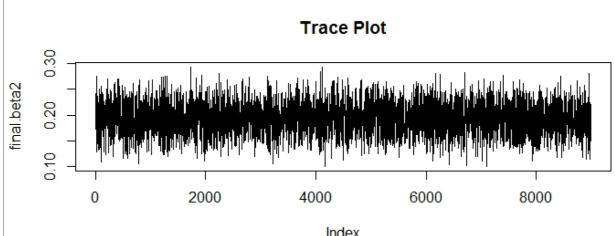
Figure 2.1: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=100




density.default(x = final.beta1)
Density
0 4 8 12
0.75 0.80 0.85 0.90 0.95
N = 9000 Bandwidth = 0.004064


Trace of var1
beta1
Density of var1
b1.mcmc
0.80 0.90
0 2000 6000
Iterations
mcmc
0 4 8 12
0.75 0.80 0.85 0.90 0.95
N = 9000 Bandwidth = 0.004786


Trace Plot
sampled.beta2
0 5 15
0 2000 4000 6000 8000 10000
Index


Trace Plot
final.beta2
0.10 0.20 0.30
0 2000 4000 6000 8000
Index

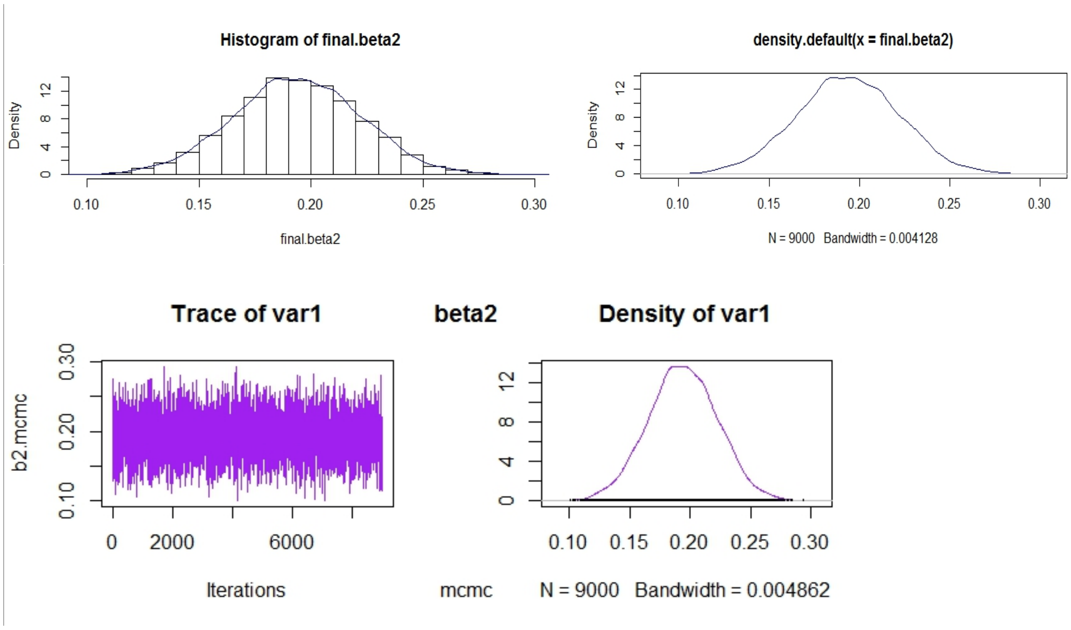
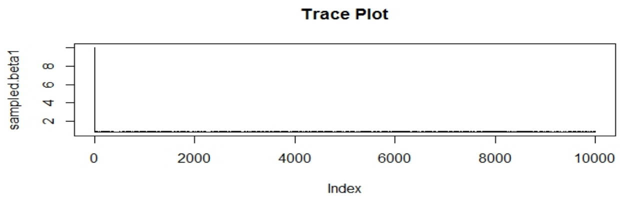
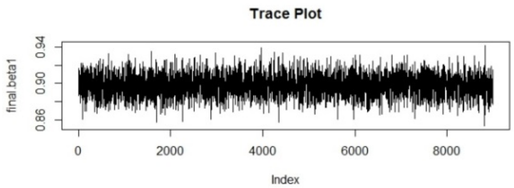
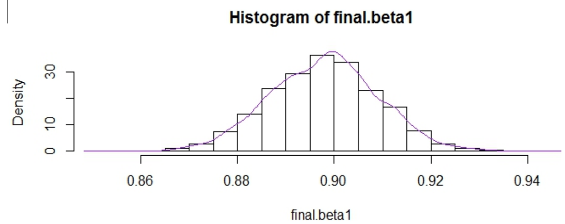
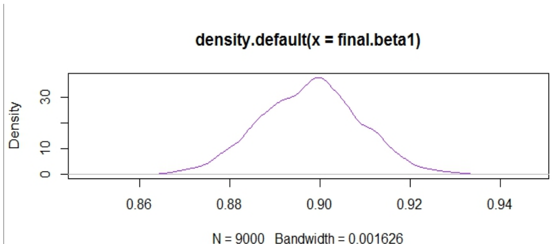


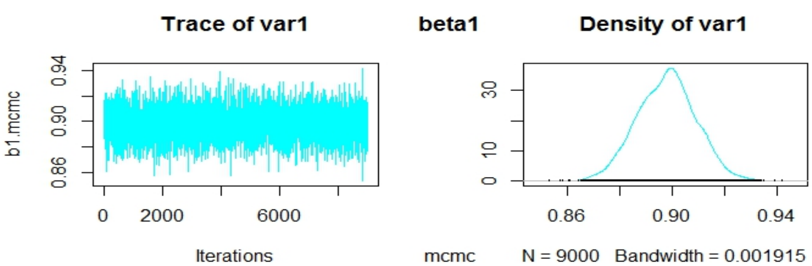
Figure 2.2: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=500

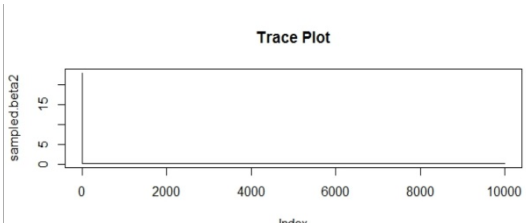

Trace Plot
sampled.beta1
Index

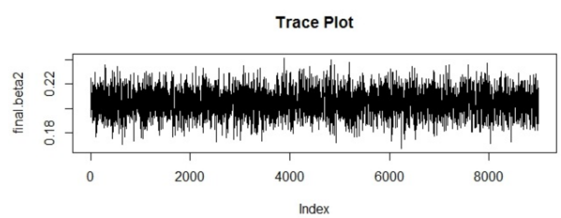

Trace Plot
final.beta1
Index

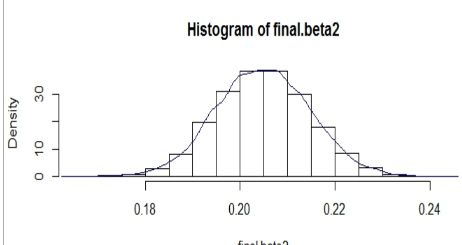

Histogram of final.beta1
Density
final.beta1

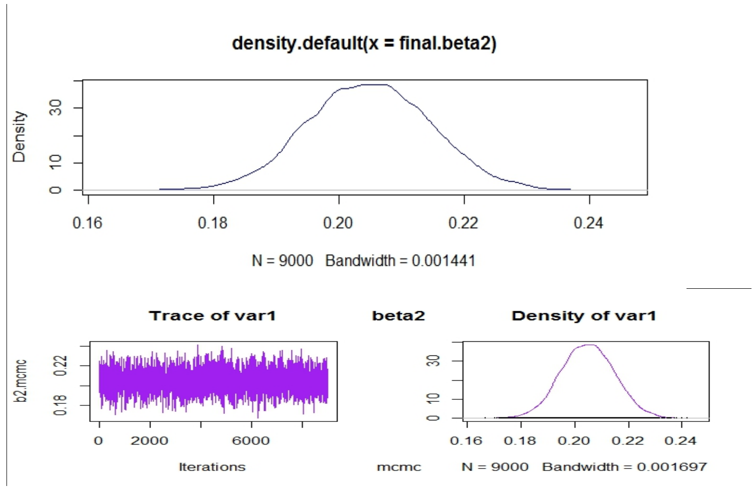

density.default(x = final.beta1)
Density
N = 9000 Bandwidth = 0.001626


Trace of var1
beta1
Density of var1
b1.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.001915


Trace Plot
sampled.beta2
Index

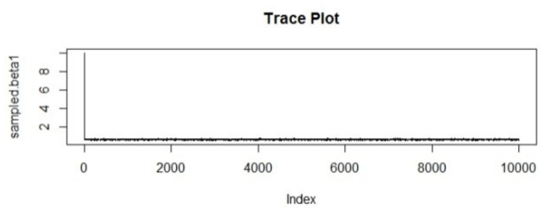

Trace Plot
final.beta2
Index

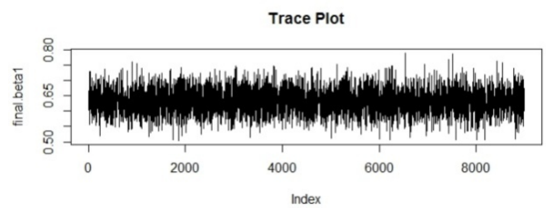

Histogram of final.beta2
Density
final.beta2

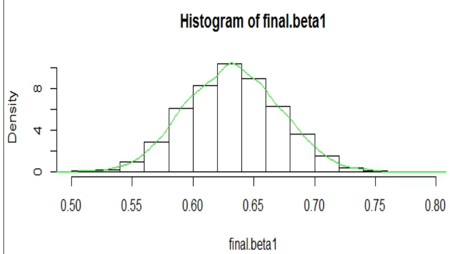


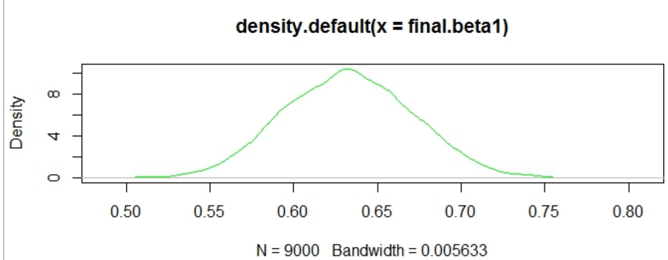
For Cobb-Douglas production with additive error term, when the production function assumed a decrease returns to scale that is ($\beta_1(0.65)$, $\beta_2(0.15)$)

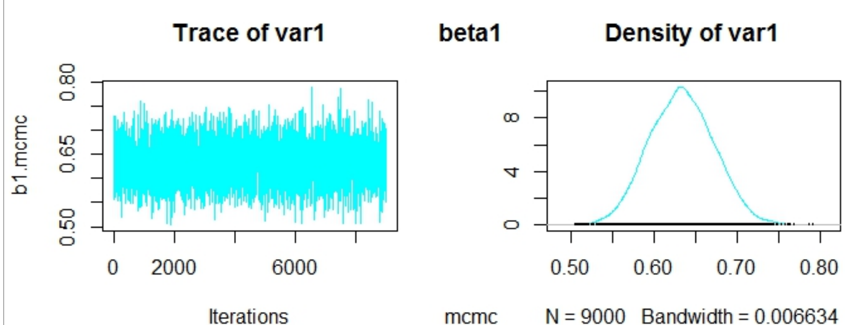
Figure 3.0: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=50

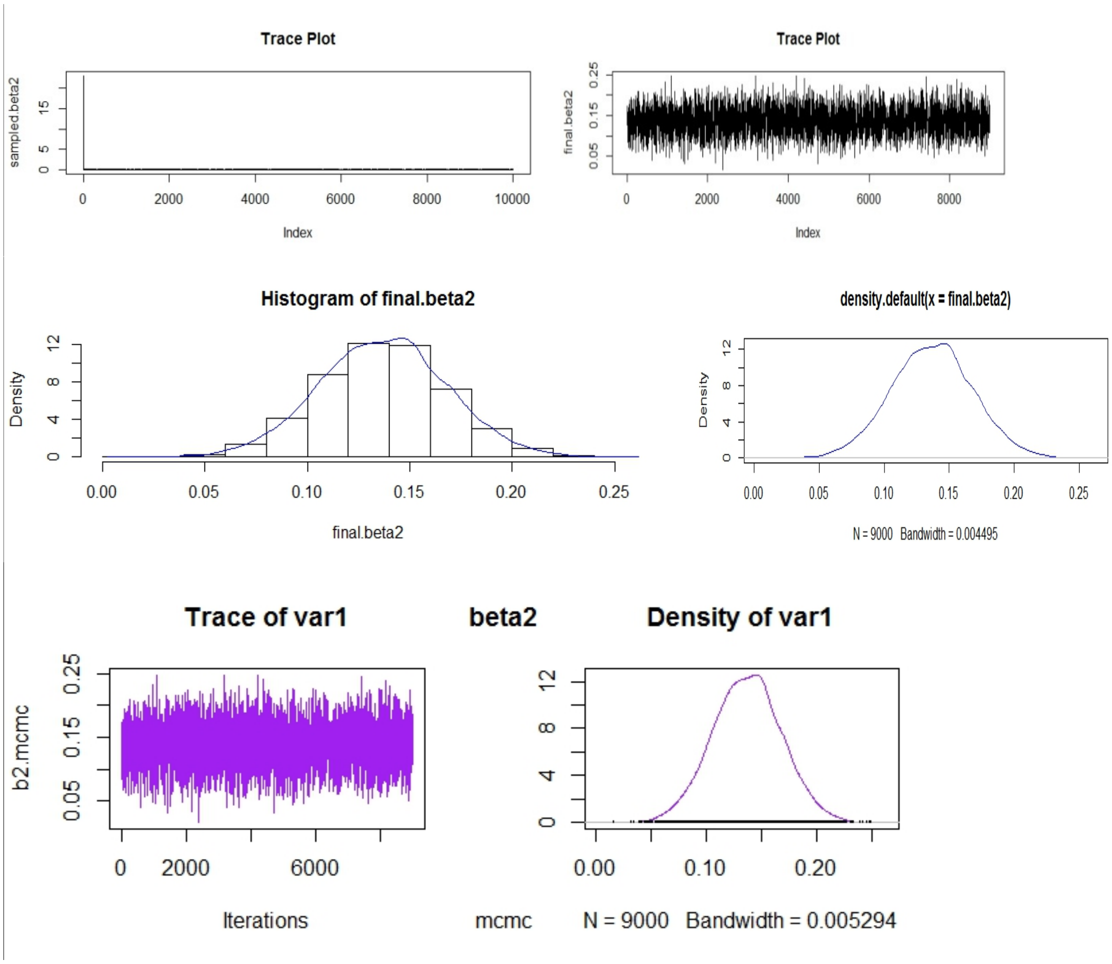

Trace Plot
sampled.beta1
Index


Trace Plot
final.beta1
Index


Histogram of final.beta1
Density
final.beta1

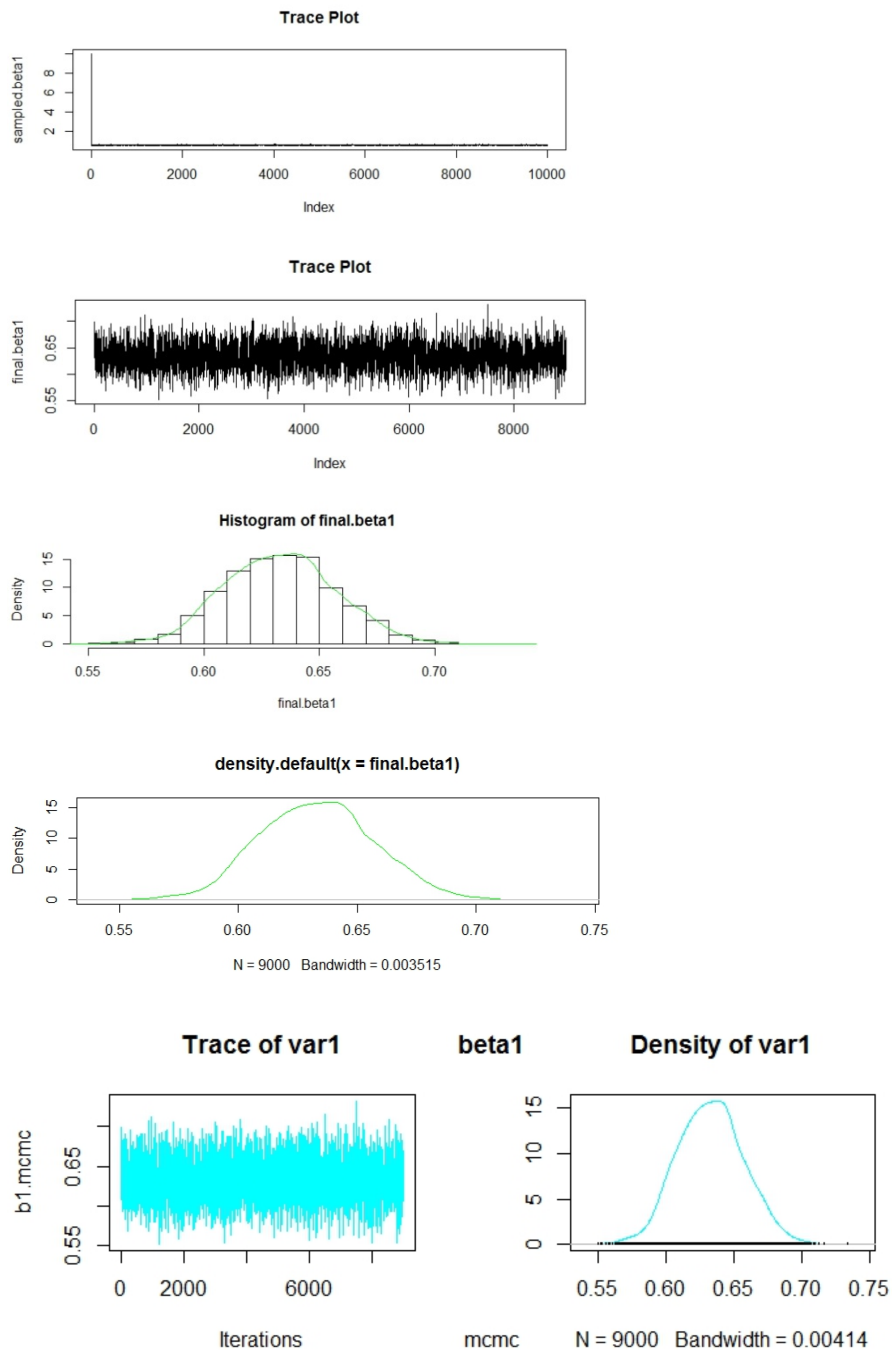

density.default(x = final.beta1)
Density
N = 9000 Bandwidth = 0.005633

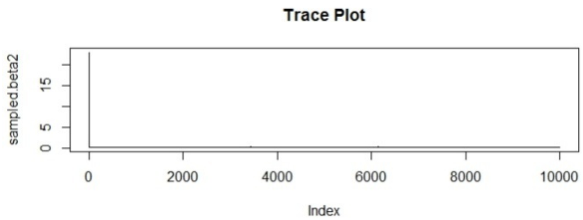

Trace of var1
beta1
Density of var1
b1.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.006634

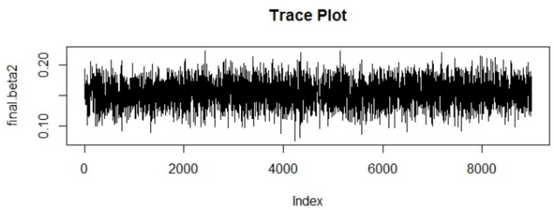


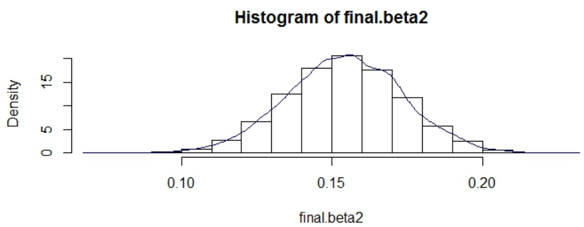
Trace Plot
sampled.beta2
Index
Trace Plot
final.beta2
Index
Histogram of final.beta2
Density
final.beta2
density.default(x = final.beta2)
Density
N = 9000 Bandwidth = 0.004495
Trace of var1
beta2
Density of var1
b2.mcmc
Iterations
mcmc
N = 9000 Bandwidth = 0.005294

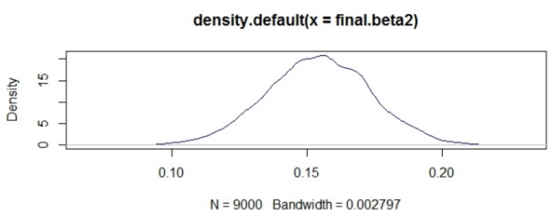
Figure 3.1: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=100




Trace Plot
sampled.beta2
0
5
15
0
2000
4000
6000
8000
10000
Index


Trace Plot
final.beta2
0.10
0.20
0
2000
4000
6000
8000
Index


Histogram of final.beta2
Density
0
5
15
0.10
0.15
0.20
final.beta2


density.default(x = final.beta2)
Density
0
5
15
0.10
0.15
0.20
N = 9000 Bandwidth = 0.002797

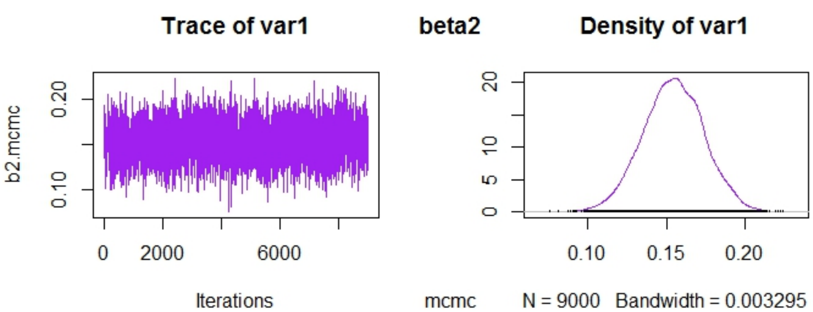
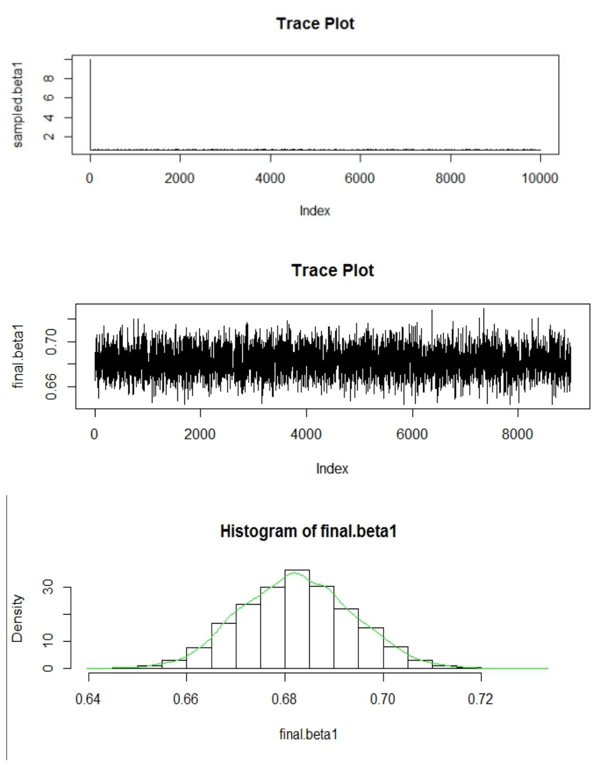
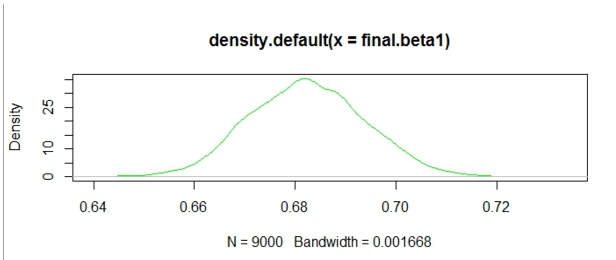
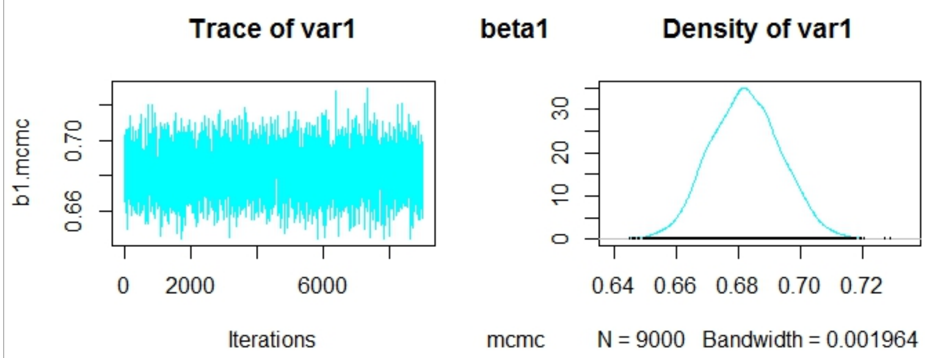
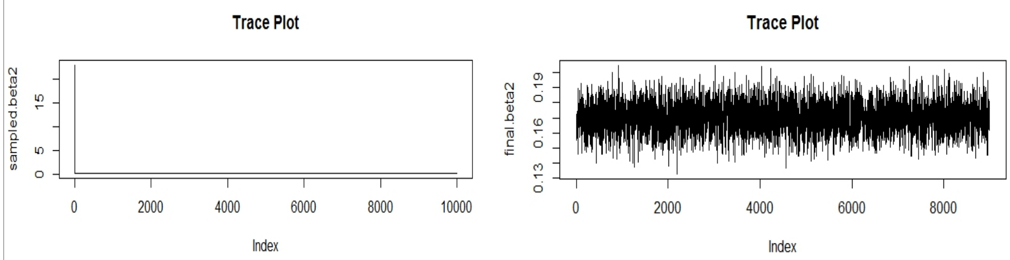

Trace of var1
beta2
Density of var1
b2.mcmc
0.10
0.20
0
2000
6000
Iterations
mcmc
0
5
10
20
0.10
0.15
0.20
N = 9000 Bandwidth = 0.003295

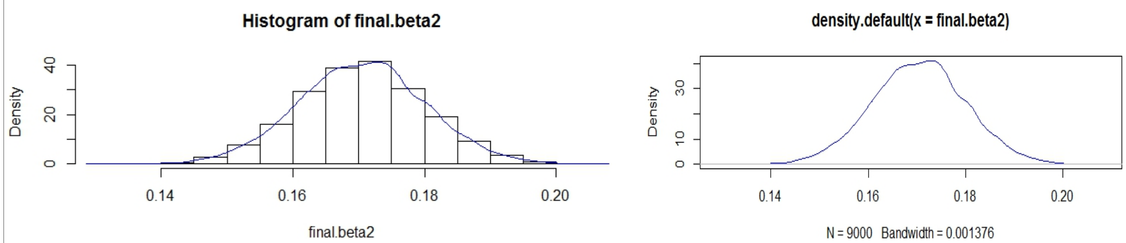
Figure 3.2: Summary of metropolis-within-Gibbs draw for β_1 and β_2 at N=500

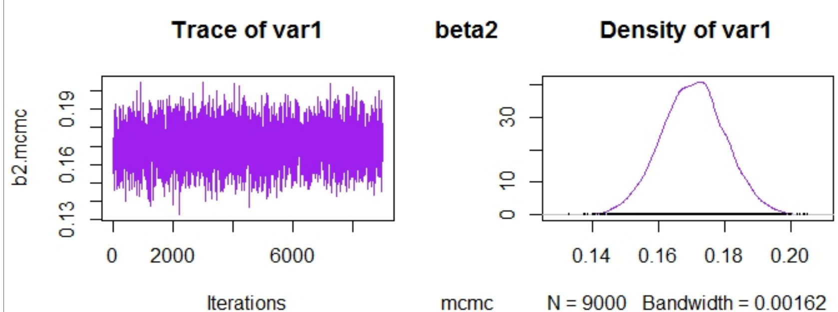



density.default(x = final.beta1)
Density
0
10
25
0.64
0.66
0.68
0.70
0.72
N = 9000 Bandwidth = 0.001668


Trace of var1
beta1
Density of var1
b1.mcmc
0.66
0.70
0
2000
6000
Iterations
mcmc
0
10
20
30
0.64
0.66
0.68
0.70
0.72
N = 9000 Bandwidth = 0.001964


Trace Plot
sampled.beta2
0 5 15
0 2000 4000 6000 8000 10000
Index
Trace Plot
final.beta2
0.13 0.16 0.19
0 2000 4000 6000 8000
Index


Histogram of final.beta2
Density
0 20 40
0.14 0.16 0.18 0.20
final.beta2
density.default(x = final.beta2)
Density
0 10 30
0.14 0.16 0.18 0.20
N = 9000 Bandwidth = 0.001376


Trace of var1
beta2
Density of var1
b2.mcmc
0.13 0.16 0.19
0 2000 6000
Iterations
mcmc
0 10 30
0.14 0.16 0.18 0.20
N = 9000 Bandwidth = 0.00162